44th International Mathematics Olympiad, Tokyo 2003

First day

1. Let S be the set of the first 1000000 non-zero positive integers, and let $A \subset S$ such that A has 101 elements. Prove that there exists the distinct positive integers $t_1, t_2, ..., t_{100} \in S$ such that the sets $A_j = [x + t_j | x \in A]$ for all $j = 1, 2, 3..., 100$ are pairwise disjoint.

2. Find all positive integers (a, b) such that the number

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is also a positive integer.

3. Given is a convex hexagon with the property that the segment connecting each pair of opposite sides is $\frac{\sqrt{3}}{2}$ times the sum of those sides. Prove that the hexagon has all angles equal to 120 degrees.

Second Day

1. Given is a cyclic quadrilateral $ABCD$ and let P, Q, R be feet of the altitudes from D to AB, BC and CA respectively. Prove that if $PR = RQ$ then the interior angle bisectors of the angles $\widehat{ABC}$ and $\widehat{ADC}$ are concurrent on AC.

2. Let $n \in N, n \geq 2$ and $x_1 \leq x_2 \leq ... \leq x_n$ be real numbers. Prove that

$$(\sum |x_i - x_j|)^2 \leq \frac{2}{3}(n^2 - 1) \sum (x_i - x_j)^2$$

Prove that this inequality is equality if and only if the sequence $(x_n)_{n \geq 1}$ is an arithmetical progression.

3. Prove that for each given prime p there exists a prime q such that $n^p - p$ is not divisible by q for each positive integer n.