Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii clasa a III-a

I. Suma este 351. Variante:

105 + 246

146 + 205

106 + 245

145 + 206

Barem: Aflarea sumei: 5p
Pentru fiecare varianta scrisa: 1p
Din oficiu 1 p.

II. **a)** $100-(30+50)-(2+10)=8$ (4p)

b) $C=5$; $A=3$; $B=6$

$365+65+5=435$ (5p)

III. Se calculeaza suma grupându-se termenii astfel:

$(3+17)+(5+15)+(7+13)+(22+38)+(24+36)+(26+34)=20+20+20+60+60+60=240$

(Calculul sumei: 3p)

$240-210=30$ (3p)

$30:2=15$ (1p)

Asadar, semnul „–” se pune în fata numarului 15. (finalizare: 2p)

1p. din oficiu.

IV. Variante de punctaj posibile:

			Total
10	10	10	30p
10	10	5	25p
10	5	5	20p
10	10	1	21p
10	1	1	12p
10	5	1	16p
5	5	5	15p
5	5	1	11
5	1	1	7p
1	1	1	3p

(0,5p pentru fiecare caz) (5 puncte)

Se constata ca: $21=7\times 3$ (2 puncte)

Primul candidat a obtinut 21 puncte iar al doilea 7 puncte (1 punct).

Diferenta de punctaj: $21-7=14p$ (1 punct)

(1 punct din oficiu)

Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii clasa a IV-a

I. **a)** $A = 140$, $B = 7$

b) Grupam:

$(3-2)+(5-4)+(7-6)+\ldots+(2007-2006) = \underbrace{1+1+1+\ldots+1+1}_{de\ 1003 ori} = 1003.$

II. Numerele cautate sunt:

158, 257, 356, 455, 554, 653, 752 si 851. (câte 1p pentru fiecare)

Cel mai mare este 851. (1p)

III. Daca la 5 gaini corespund 2 rate, atunci la 15 gaini corespund 6 rate, iar la 6 rate corespund 2 gâste. Într-o astfel de grupa formata sunt:

$15+6+2=23$ pasari.

$92:23=4$ grupe

Asadar, $4\times 15=60$ gaini

$4\times 6=24$ rate

$4\times 2=8$ gâste

Barem: Stabilirea unei grupe: 6p

Stabilirea grupelor: 1p

Stabilirea nr. de pasari de fiecare fel câte 2p.

1p din oficiu

IV. Relatiile: $\overline{xn}+\overline{my}=81$ si

$\overline{zv}+\overline{ut}=125$

se mai pot scrie astfel:

$$\begin{cases}10x+n+10m+y=81\\10z+v+10u+t=125\end{cases}$$ (1p)

$$\begin{cases}10x+y+10m+n=81\\10z+t+10u+v=125\end{cases}$$ (1p)

$$\begin{cases}\overline{xy}+\overline{mn}=81\,|\times 100\\\overline{zt}+\overline{uv}=125\end{cases}$$ (2p)

$$\begin{cases} \overline{xy}\times 100+\overline{mn}\times 100=8100 \\ \overline{zt}+\overline{uv}=125 \end{cases}$$ (2p)

si adunam relatiile:

$\overline{xy}\times 100+\overline{zt}+\overline{mn}\times 100+\overline{uv}=8225$ (1p)

care se scrie:

$\overline{xyzt}+\overline{mnuv}=8225$ (1p)

Adica $\left(\overline{xyzt}+\overline{mnuv}\right):5=1645$ (1p)

1 p din oficiu.

Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii clasa a V-a

I. **a)** $2+2\cdot[22+2\cdot(222-222)]=2+2\cdot(22+2\cdot 0)=2+2\cdot 22=2+44=46$

b) $135\cdot(75-65)+65\cdot(124-114)=135\cdot 10+65\cdot 10=10\cdot(135+65)=10\cdot 200=2000$

c) $401-33=368$

II. **1)** Fie x si y cele doua numere $\Rightarrow x=y+2006 \Rightarrow x-y=2006$. (1p)

Dar $x+y=2\cdot(x-y)+2 \Rightarrow$ (1p)

$x+y=2\cdot 2006+2=4014 \overset{(1p)}{\Rightarrow} y+2006+y=4014 \Rightarrow 2y=4014-2006 \Rightarrow 2y=2008 \Rightarrow$

$y=1004$ (1p) $\Rightarrow$ $x=3010$. (1p)

2) $a\cdot 1111+b\cdot 111+c\cdot 11+d=2604$ (1p)

$a=1 \Rightarrow b\cdot 111+c\cdot 11+d=1493$. (0,5p)

În stânga avem maxim $9\cdot 111+9\cdot 11+9=1107<1493$ (0,5p)

$\Rightarrow$ nu avem solutii. (0,5p)

$a=2 \Rightarrow b\cdot 111+c\cdot 11+d=382$ (0,5p)

$b\le 2 \Rightarrow$ în stânga avem maxim $2\cdot 111+9\cdot 11+9=330$

$<382 \Rightarrow$ nu avem solutie. (0,5p)

$b=3 \Rightarrow c\cdot 11+d=49 \Rightarrow c=4$ si $d=5 \Rightarrow 2345+234+23+2=2604$ (A) (0,5p)

III. Daca x cifra $\Rightarrow$ y,z cifre $\Rightarrow$ $60=x+y+z\le 27$ fals. (1p)

$\Rightarrow x=\overline{ab} \Rightarrow y=a+b$

a) Daca $a+b<10 \Rightarrow z=a+b$ (1p)

b) Daca $a+b\ge 10$, cum $a+b\le 18 \Rightarrow z=\overline{1c}$ iar $c=a+b-10$. (1p)

a) $a\cdot 10+b+a+b+a+b=60 \Rightarrow a\cdot 12+b\cdot 3=60\,|:3 \Rightarrow 4a+b=20$ (1p)

Daca $a=1$ sau $a=2$ $\Rightarrow$ nu avem solutii. (0,5p)

$a=3 \Rightarrow b=8$, dar $3+8>10$ nu convine. (0,5p)

$a=4 \Rightarrow b=4 \Rightarrow x=44,\ y=8,\ z=8$ (0,5p)

$a=5 \Rightarrow b=0 \Rightarrow x=50,\ y=5,\ z=5$ (0,5p)

b) $a\cdot 10+b+a+b+1+a+b-10=60 \Rightarrow$

$12a+3b=69\,|:3 \Rightarrow 4a+b=23 \Rightarrow a=4$ si $b=7 \Rightarrow x=47,\ y=11,\ z=2$. (2p)

Sau $a=5 \Rightarrow b=3$ $a+b=8<10$ nu convine. (1p)

1p din oficiu

IV. $\left.\begin{array}{l} A-B\geq 10 \\ A+B\geq 1000 \end{array}\right| \Rightarrow$ (3p) $A+B+A-B\geq 1010 \Rightarrow 2A\geq 1010 \Rightarrow A\geq 505$ (1p)

Pentru $A\geq 505$ iau $B=A-10 \Rightarrow$ (1p) $\Rightarrow$

$A-B=A-(A-10)=10$ si $A+B=A+A-10=2A-10\geq 2\cdot 505-10 \Rightarrow A+B\geq 1000$

Deci numerele sunt 505, 506, …, 999 (2p) $\Rightarrow$ în total $999-504=495$ numere. (2p)

1p din oficiu.

Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii clasa a VI-a

I. **1)** $a = (100+81+64):49+(25-9):4 = 245:49+16:4 = 5+4 = 9$ (2p)

$b = 200+(81+27):9+9+4 = 200+108:9+13 = 20+12+13 = 225$ (2p)

$\Rightarrow (a,b) = 9$ si $[a,b] = 225$. (1p)

2) $2^n \cdot 3^n \cdot (2^3+6^2-3) = 2^n \cdot 3^n \cdot 41 \vdots 41$. (4p)

II. **1)** $\overline{abab} = \overline{ab}\cdot 100+\overline{ab} = \overline{ab}\cdot 101$. (1p)

Ca sa avem numar minim de divizori $\Rightarrow \overline{ab}$ prim. (1p)

Cel mai mic, respectiv mare, numar prim de doua cifre este 11 respectiv 97. Numerele cautate vor fi: $101\cdot 11 = 1111$ si $101\cdot 97 = 9797$. (1p)

2) a) $\frac{3\cdot 12}{60} < \frac{5n}{60} < \frac{11\cdot 4}{60} \Leftrightarrow 36 < 5n < 44 \Leftrightarrow \frac{36}{5} < n < \frac{44}{5} \Rightarrow n = 8$. (2p)

b) $\frac{12}{7} < \frac{n}{6} < \frac{29}{14} \Leftrightarrow \frac{12\cdot 6}{7} < n < \frac{29\cdot 6}{14} \Rightarrow n \in \{11;12\}$. (2p)

3) Notam $d = (a,b)$. Deoarece $d \mid a$ si $a \mid [a,b]$ $\Rightarrow d \mid [a,b]$; $(a,b)+[a,b] = 101 \Rightarrow d + d\cdot k = 101 \Rightarrow d(1+k) = 101$ prim $\Rightarrow d = 1 \Rightarrow [a,b] = 100$. (1p)

$\Rightarrow a\cdot b = 100$ si $(a,b) = 1 \Rightarrow$ solutiile: $(1,100),(100,1),(4,25),(25,4)$. (1p)

III. **a)** $A_{19}A_{20} = 19mm$ (3p)

b) $A_1A_{20} = A_1A_2 + A_2A_3 + A_3A_4 + \ldots + A_{19}A_{20} = (1+2+3+\ldots+19)mm = 19\cdot 20:2mm = 190mm = 19cm$ (3p)

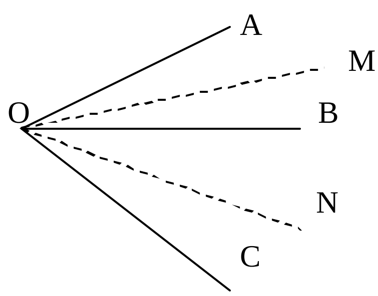
c) M mijlocul lui $[A_1A_{20}] \Rightarrow A_1M = MA_{20} = \frac{A_1A_{20}}{2} = \frac{190mm}{2} = 95mm$ (1p)

N mijlocul lui $[A_2A_{19}] \Rightarrow A_2N = NA_{19} = \frac{A_2A_{19}}{2} = \frac{A_1A_{19}-A_1A_2}{2} = \frac{18\cdot 19:2-1}{2} = \frac{171-1}{2} = \frac{170}{2} = 85$(mm)

$A_1N = A_1A_2 + A_2N = 1+85 = 86$ (mm) (1p)

$MN = A_1M - A_1N = 95mm - 86mm = 9mm$. (1p)

IV. **a)**

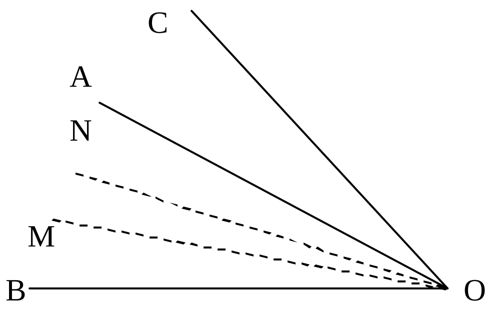


$m(\angle MON) = \dfrac{m(\angle AOB) + m(\angle BOC)}{2} \Rightarrow$

$2\overline{ac} = \overline{ab} + \overline{bc} \Rightarrow 20a + 2c = 10a + 11b + c \Rightarrow$

$10a + c = 11b \Rightarrow 10(a-b) = b - c \Rightarrow 10 \mid b - c \Rightarrow b - c = 0 \Rightarrow b = c$ nu convine. (1p)

b)



$m(\angle MON) = \dfrac{m(\angle BOC) - m(\angle AOB)}{2} \Rightarrow$

$\Rightarrow 2\overline{ac} = \overline{ab} - \overline{ab} \Rightarrow 2\overline{ac} + \overline{ab} = \overline{bc} \Rightarrow 30a + 2c + b = b \cdot 10 + c \Rightarrow 30a + c = 9b \Rightarrow$

$c \,\vdots\, 3 \Rightarrow c \in \{3,6,9\}$.

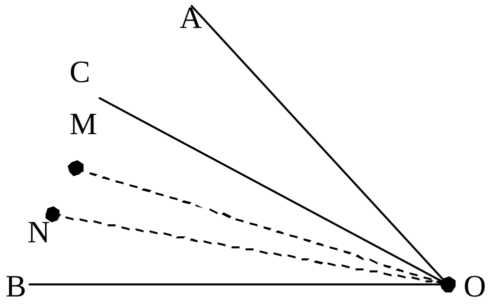
$c = 3 \Rightarrow 30a + 3 = 9b \Rightarrow 10a + 1 = 3b \Rightarrow a = 2,\ b = 7$

$c = 6 \Rightarrow 30a + 6 = 9b \Rightarrow 10a + 2 = 3b \Rightarrow a = 1,\ b = 4$

$c = 9 \Rightarrow 30a + 9 = 9b \Rightarrow 10a + 3 = 3b \Rightarrow$ imposibil

Solutiile sunt $\{2,7,3\}$ si $\{1,4,6\}$. (2p)

c)



Deoarece $m(\angle MON) < m(\angle BOM) = \dfrac{M(\angle AOB)}{2} \Rightarrow$

$2\overline{ac} < \overline{ab} \Leftrightarrow 20a + 2c < 10a + b \Leftrightarrow 10a + 2c < b$ fals. (1p)

2) $m(\angle AOC) = M(\angle BOC) - m(\angle AOB) = \overline{bc} - \overline{ab}$ (1p)

a) $a = 2,\ b = 7,\ c = 3$.

$\Rightarrow m(\angle AOC) = 73^\circ - 27^\circ = 46^\circ$ (2p)

b) $a = 1,\ b = 4,\ c = 6$

$\Rightarrow m(\angle AOC) = 46^\circ - 14^\circ = 32^\circ$. (2p)

Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii clasa a VII-a

I. **a)** $a=15 \Rightarrow A=(1-30)(2-30)....(15-30)=-15\cdot 16\cdot 17...29$

$B=(-1)^{1+2+...+15}=1; \quad C=-1$

$A<C<B.$ (2p)

ii) $A=0;$ $B=(-1)^{1+2+...+2007}=1;$ $C=(-1+1)...(-1)^{2007}=-1$

$C<A<B$ (3p)

b) $\frac{1}{6}+\frac{1}{13}=\frac{19}{78}$

$n=0,2(435897)(a_8.....a_{13})(a_{14}.......a_{19})........... \Rightarrow a_{2006}=4$ (2p)

$2005=6\cdot 334+1 \Rightarrow$

$a_1+a_2+...+a_{2006}=2+(a_2+.......a_7)+(........)........+4=$

$=2+334(4+3+5+8+9+7)+4=6+334\cdot 36=12030$ (2p)

II. **a)** Avem $\begin{matrix} n+2 \mid n^2+3 \\ n+2 \mid n(n+2) \end{matrix} \Rightarrow n+2 \mid n^2+2n-n^2-3$

$\Rightarrow n+2 \mid 2n-3$, dar $n+2 \mid 2n+4 \Rightarrow$ (2p)

$\Rightarrow n+2 \mid 7 \Rightarrow n+2 \in \{-7;-1;1;7\} \Rightarrow n \in \{-9;-3;-1;5\}$ (2p)

b) Daca $a<0 \Rightarrow \begin{matrix} a+a^3+a^5<0 \\ a^2+a^4+a^6>0 \end{matrix}$

$\Rightarrow a+a^3+a^5 \neq a^2+a^4+a^6$ (1p)

Daca $a\in(0,1) \Rightarrow \left.\begin{matrix} a>a^2 \\ a^3>a^4 \\ a^5>a^6 \end{matrix}\right| \Rightarrow a+a^3+a^5 \neq a^2+a^4+a^6$ (2p)

Daca $a\in(1,\infty) \Rightarrow \left.\begin{matrix} a^2>a \\ a^4>a^3 \\ a^6>a^5 \end{matrix}\right| \Rightarrow a^2+a^4+a^6 \neq a+a^3+a^5$ (1p)

Daca $\begin{matrix} a=0 \Rightarrow 0=0 \\ a=1 \Rightarrow 1=1 \end{matrix}$ verifica egalitatea (1p)

III. **a)** Triunghiurile ABE si ECD sunt isoscele.

Fie $a = mas\angle BAE = mas\angle BEA$ si

$b = mas\angle CED = mas\angle CDE$ (1p)

$mas\angle B = 180^{\circ} - 2a$ (1p)

$mas\angle C = 180^{\circ} - 2b$ (1p)

$\hat{B} + \hat{C} = 360 - 2(a + b) = 360^{\circ} - 180^{\circ} = 180^{\circ}$. Conform reciprocei teoremei lui Euclid

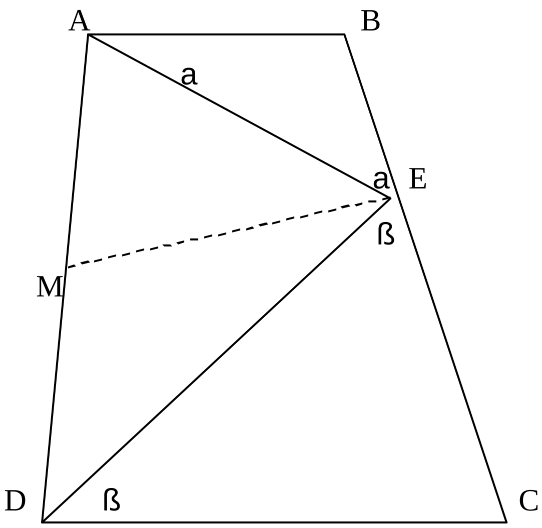
$\Rightarrow AB \| CD$ (2p)

b) În $tr.AED$, ME este mediana, deci $[ME] \equiv [MA] \equiv [MD]$. (1p)

De aici rezulta $\Delta BAM \equiv \Delta BEM$ si $\Delta CEM \equiv \Delta CDM$ (1p) deci $mas\angle BME = \frac{1}{2} mas\angle AME$ si

$mas\angle CME = \frac{1}{2} mas\angle DME$. (1p)

Deaici rezulta ca $mas\angle BMC = 90^{\circ}$. (1p)



IV. Notam $\angle BAC = a$, $AB = AC = b$, $BC = a$, N mijlocul laturii AC.

i) Daca ΔABC este ascutitunghic nu putem avea decât ordinea $AMC'B$. Ordinea $AC'MB$ apare daca $a > b$, dar în acest caz avem $2 \cdot C'M < b$. (1p)

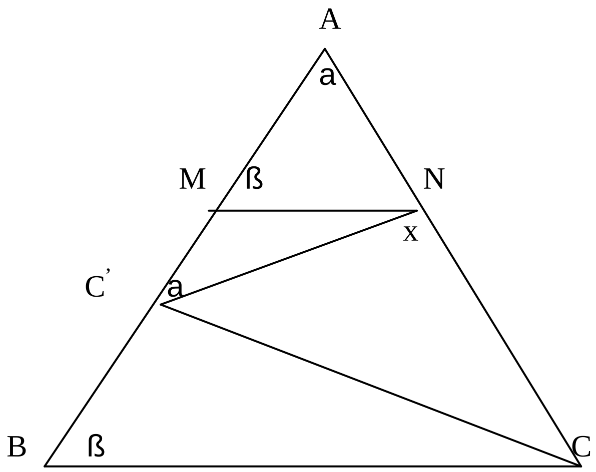
Deoarece $MN = \frac{BC}{2} = C'M$, triunghiul $C'MN$ este isoscel (cu $\angle MC'N = \angle MNC'$). $C'N$ este mediana în triunghiul dreptunghic $AC'C$, deci si triunghiul $AC'N$ este isoscel. (1p)

Avem: $\angle MC'N = a = \angle MNC'$ si cum $\angle AMN$ este unghi exterior obtinem:

$\angle AMN = 2a$

Dar $MN \| BC$ si deci $B = 2\alpha$ (1p)

Obtinem $\alpha + 2\alpha + 2\alpha = 180^\circ \Rightarrow \alpha = 36^\circ$, adica $A = 36^\circ$, $B = C = 72^\circ$ (1p)



ii) Daca ΔABC este obtuzunghic, avem ordinea $C'AMB$. (1p)

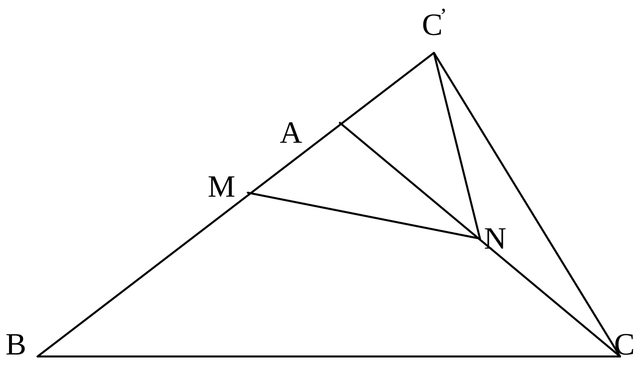
Urmarind un rationament ca mai sus obtinem:

$\angle AC'N = \angle C'AN = 180^\circ - \alpha$ (1p)

$\angle MNC' = \angle MC'N = 180^\circ - \alpha$ (1p)

$\angle ABC = \angle C'MN = 180^\circ - 2(180^\circ - \alpha) = 2\alpha - 180^\circ$ (1p)

Rezulta $\alpha + 2(2\alpha - 180^\circ) = 180^\circ$, adica $A = 108^\circ$, $B = C = 36^\circ$. (1p)



Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii clasa a VIII-a

I. **a)** Daca $\sqrt{\dfrac{\overline{xy}+12}{\overline{xy}-12}} \in N$ atunci $\dfrac{\overline{xy}+12}{\overline{xy}-12} = k^2$ (1p)

$k \in N \Rightarrow \dfrac{\overline{xy}+12}{\overline{xy}-12} = 1 + \dfrac{24}{\overline{xy}-12} \Rightarrow$ (1p)

$\left.\begin{array}{l}\overline{xy}-12 \in \{1,2,3,4,6,8,12,24\} \Rightarrow \\ \overline{xy} \in \{13,14,15,16,18,20,24,36\}\end{array}\right\}$ (1p)

$\overline{xy} = 13 \Rightarrow \dfrac{13+12}{13-12} = 25 = 5^2\,(A)$

$\overline{xy} = 14 \Rightarrow \dfrac{14+12}{14-12} = 13(F)$

$\overline{xy} = 15 \Rightarrow \dfrac{15+12}{15-12} = \dfrac{27}{3} = 9 = 3^2\,(A)$

$\overline{xy} = 16 \Rightarrow \dfrac{16+12}{16-12} = \dfrac{28}{4} = 7(F)$

$\overline{xy} = 18 \Rightarrow \dfrac{18+12}{18-12} = 5(F)$ (1p)

$\overline{xy} = 20 \Rightarrow \dfrac{20+12}{20-12} = 4(A)$

$\overline{xy} = 24 \Rightarrow \dfrac{24+12}{24-12} = 3(F)$

$\overline{xy} = 36 \Rightarrow \dfrac{36+12}{36-12} = 2(F)$

Asadar, $\overline{xy} \in \{13,15,20\}$ (1p)

b) $(x+y-2)^2 \ge 0$, $\sqrt{z-x-y} \ge 0$, $|2x-y+3| \ge 0 \Rightarrow$ (1p)

$x+y-2=0$, $z-x-y=0$, $2x-y+3=0$ (1p)

$\left.\begin{array}{l}x+y-2=0\\ 2x-y+3=0\end{array}\right\} \Rightarrow \begin{cases}x+y=2\\ 2x-y=-3\end{cases} \Rightarrow 3x=-1$

$\Rightarrow x = -\dfrac{1}{3} \Rightarrow \quad y \overset{3)}{=} 2 + \dfrac{1}{3} = \dfrac{7}{3}$

$z-x-y=0 \Rightarrow z = x+y = -\dfrac{1}{3} + \dfrac{7}{3} = \dfrac{6}{3} = 2$. (1p)

II. **a)** Din $ab \in Q$ si $\frac{a}{b} \in Q \Rightarrow ab \cdot \frac{a}{b} = a^2 \in Q$.

Analog $b^2 \in Q$ (1p)

Avem:

$a(a^3 + b^3) = a^4 + ab^3 = (a^2)^2 + (ab)b^2 \in Q$ (2p)

Dar $a^3 + b^3 \in Q \Rightarrow a \in Q$. (1p)

Analog $b \in Q$.

1p din oficiu

b) Avem:

$X = 6a^3 + (a^3 + 3b^3 + 2a)$; $Y = 6b^3 - (b^3 + 2b - 3a^2)$

Este suficient sa aratam ca $X + Y \vdots 6$ (2p)

Avem: $X + Y = 6(a^3 + b^3) + a^3 + 3a^2 + 2a - (b^3 + 2b - 3b^2)$

Dar $a^3 + 3a^2 + 2a = a(a^2 + 3a + 2) = a(a+1)(a+2) \vdots 6$ (1p)

$b^3 - 3b^2 + 2b = b(b^2 - 3b + 2) = b(b-1)(b-2) \vdots 6$ (1p)

Deci $X + Y \vdots 6$

III. **a)** (OM) este linie mijlocie în $\Delta ACB'$ $\Rightarrow OM \parallel AB'$

Cum $AB' \subset (AB'D') \Rightarrow OM \parallel (AB'D')$ (1p)

b) (OM) si AD' sunt drepte necoplanare, dar $OM \parallel AB' \Rightarrow \angle(OM, AD') \equiv \angle(AB', AD')$ (1p)

$\Delta AB'D'$ este echilateral, deoarece $AB' = B'D' = AD' = l\sqrt{2}$ (l fiind lungimea laturii cubului) $\Rightarrow$ $m(B'\hat{A}D') = 60^\circ \Rightarrow m(\angle(OM, AD')) = 60^\circ$

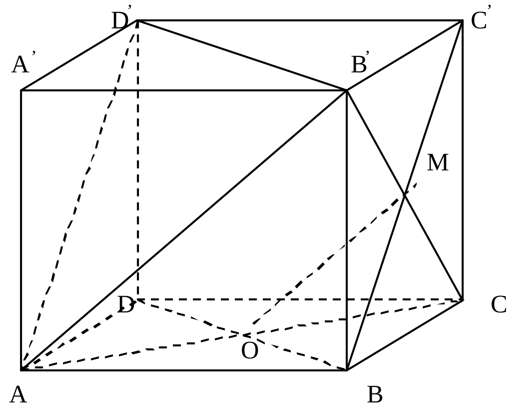
c) $MO \subset (BMD)$, $MO \parallel AB'$ si $AB' \subset (AB'D') \Rightarrow MO \parallel (AB'D')$ (1p)

$BB' \parallel DD'$ si $[BB'] \equiv [DD']$ (ca muchii ale cubului) $\Rightarrow$ $BDD'B'$ paralelogram

$BB' \perp (ABC) \Rightarrow BDD'B'$ dreptunghi) $\Rightarrow$ $BD \parallel B'D'$.

Dar $B'D' \subset (AB'D') \Rightarrow BD \parallel (AB'D')$ (2p)

Din (1) si (2), cum OM si BD sunt concurente din planul (DMB') $\Rightarrow$ $(DMB) \parallel (AB'D')$ (2p)



IV. a) $\Rightarrow$ b)

$$\left.\begin{array}{l}(MNP)\|(A'B'C')\\ (A'B'C')\cap(A'C'B)=A'C'\\ (MNP)\cap(A'C'B)=MN\end{array}\right\}\Rightarrow MN\|A'C'$$ (1p)

$$\left.\begin{array}{l}(ACC'A')\cap(ACNM)=AC\\ MN\|A'C'\end{array}\right\}\Rightarrow A'C'\|AC\|MN$$ (Teorema acoperisului) (1p)

$$\left.\begin{array}{l}(MNN'M')\cap(AM'N'C)=M'N'\\ AC\|MN\end{array}\right\}\Rightarrow M'N'\|AC$$ (Teorema acoperisului) (1p)

Analog $M'P'\|BC$, $N'P'\|AB$ $\Rightarrow$ (din T. Thales)

$$\left.\begin{array}{l}\dfrac{AM'}{M'B}=\dfrac{CN'}{N'B}\\ \dfrac{AM'}{M'B}=\dfrac{AP'}{P'C}\end{array}\right\}\Rightarrow\left.\begin{array}{l}\dfrac{CN'}{N'B}=\dfrac{AP'}{P'C}\\ \dfrac{AP'}{P'C}=\dfrac{BN'}{N'C}\end{array}\right\}\Rightarrow\dfrac{CN'}{N'B}=\dfrac{BN'}{N'C}\Rightarrow$$

$\dfrac{CN'}{BC}=\dfrac{BN'}{BC}\Rightarrow CN'\equiv BN'$. Analog $AM'\equiv M'B$ si $AP'\equiv P'C$

$\Rightarrow N', M', P'$ mijloacele laturilor (1p)

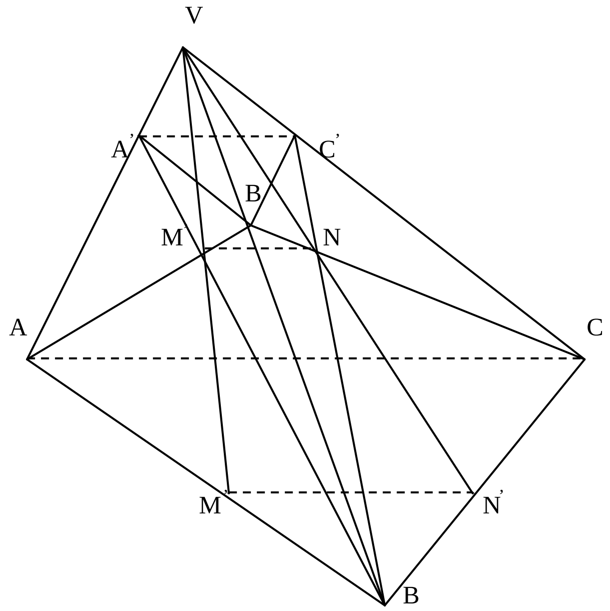
$b\Rightarrow a$

$$\left.\begin{array}{l}M'N'\|AC\\ (ACNM)\cap(M'N'NM)=MN\end{array}\right\}\Rightarrow M'N'\|AC\|MN$$

$$\left.\begin{array}{l}(ACC'A')\cap(MNC'A')=A'C'\\ AC\|MN\end{array}\right\}\Rightarrow A'C'\|MN$$

Analog $A'B'\|NP\Rightarrow(A'B'C')\|(MNP)$ (1p)

(1 p din oficiu)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Concursul de matematica Arhimede

Editia a IV-a. Etapa I-a 25 noiembrie 2006.

Solutii. Clasa a IX-a

I.

1) Avem: $x=\left(\frac{x+1}{2}\right)^2-\left(\frac{x-1}{2}\right)^2$ $(\forall)$ $x\in Q$ (4p)

2) Putem alege $x=a+1$, $y=-a$, $z=-1$. (5p)

II. **a)** Din $x+y+z\geq 3\Rightarrow (x+y+z)^2\geq 9$

Si cum $3(x^2+y^2+z^2)\geq (x+y+z)^2$ $\Rightarrow x^2+y^2+z^2\geq 3$ (2p)

Avem deci:

$x^2+y^2+z^2\geq 3$

$x^2+y^2+z^2\leq 3$ (prin ipoteza) $\Rightarrow x^2+y^2+z^2=3$ (2p)

b) Cum $x,y,z\in R^*$ $\Rightarrow$ $(x^2<3; y^2<3; z^2<3)$ (1p)

Utilizam inegalitatea: $a,b,c>0$ $\Rightarrow$ $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$ (2p)

cu $a=3-x^2$, $b=3-y^2$, $c=3-z^2$.

Adica:

$[9-(x^2+y^2+z^2)]\cdot\left(\frac{1}{3-x^2}+\frac{1}{3-y^2}+\frac{1}{3-z^2}\right)\geq 9$

Si cum $x^2+y^2+z^2=3$

$\Rightarrow \frac{1}{3-x^2}+\frac{1}{3-y^2}+\frac{1}{3-z^2}\geq\frac{9}{6}=\frac{3}{2}$ (1p)

OBSERVATIE: Egalitate avem când $x=y=z=1$ (1p)

III. **1)** Presupunem prin absurd ca $x\in R\setminus Q$. Deci $x\neq 0$. (1p)

Avem $x^{2n+1}=\frac{x^{(n+1)^2}}{x^{n^2}}\in Q$ si prin urmare $x^{2n^2+n}=\left(x^{2n+1}\right)^n\in Q$. (0,5p)

Deoarece $x^{2n^2}=\left(x^{n^2}\right)^2\in Q$ rezulta $x^n=\frac{x^{2n^2+n}}{x^{2n^2}}\in Q$. (*) (0,5p)

Din (*) rezulta $x = \dfrac{x^{2n+1}}{\left(x^n\right)^2} \in Q$. CONTRADICTIE.

Deci $x \in Q$ (1p)

2) Putem alege $x = \sqrt{2}$ si tinem seama ca $\dfrac{n^2+n}{2} \in N$ $(\forall)$ $n \in N$. (1+2p)

3) Avem $u = x^{an^2+bn+c} \in Q$ si $v = x^{a(n+1)^2+b(n+1)+c} \in Q$. (1p)

Daca $x \in R \setminus Q$ atunci $x^{a(2n+1)+b} = \dfrac{v}{u} \in Q$ si prin urmare: $x^{a(2n+3)+b} \in Q$. (0,5p)

Rezulta $x^{2a} \in Q$ si $x^{a+b} = \dfrac{x^{a(2n+1)+b}}{\left(x^{2a}\right)^n} \in Q$. (0,5p)

Din $(a+b, c) = 1$ rezulta ca exista $p, q \in Z$ a.î. (0,5p)

$p(a+b) + qc = 1$. Deci $x = x^{p(a+b)+qc} = \left(x^{a+b}\right)^p \cdot \left(x^c\right)^q \in Q$. (0,5p)

1 punct din oficiu

IV. Sa notam cu a_0 valoarea maxima a parametrului $a > 0$ pentru care are loc inegalitatea din enunt.

Înlocuind în inegalitatea din enunt $a = b = c$ obtinem $6a \ge 3aa_0$, de unde rezulta $a_0 \le 2$. (3p)

În cele ce urmeaza ne propunem sa demonstram inegalitatea:

$$\sum \frac{b^2+c^2}{a} \ge 2\sqrt{3\sum a^2} \qquad (*) \qquad (1p)$$

Daca am demonstra inegalitatea (*) atunci va rezulta ca $a_0 \ge 2$, ceea ce este echivalent cu $a_0 = 2$.

Observam ca $\sum \dfrac{b^2+c^2}{a} \ge 2\sum \dfrac{bc}{a} = 2\dfrac{\sum b^2c^2}{abc}$ (1p)

Vom demonstra ca

$$2\frac{\sum b^2c^2}{abc} \ge 2\sqrt{3\sum a^2} \qquad (**) \qquad (1p)$$

Observam ca inegalitatea (**) este echivalenta cu inegalitatea

$$\left(\sum b^2c^2\right)^2 \ge 3a^2b^2c^2\sum a^2 \qquad (***) \qquad (1p)$$

Notam $bc = x$, $ab = y$, $ac = z$. (1p) Atunci

$\left(\sum b^2c^2\right)^2 = \left(\sum x^2\right)^2 = \sum x^4 + 2\sum x^2y^2 \ge \sum x^2y^2 + 2\sum x^2y^2 = 3\sum x^2y^2 = 3a^2b^2c^2\left(\sum a^2\right)$. (2p)

1punct din oficiu

Concursul de matematica Arhimede

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Solutii. Clasa a X-a

I. **a)** Sa presupunem prin absurd ca f este monotona. Atunci se verifica usor ca functia $g(x) = f(f(x))$, $x \in M$ este crescatoare. (1p)

Din relatia $f(x) = f(f(x)) + x$ $(\forall)$ $x \in M$ rezulta ca f este strict crescatoare si prin urmare functia $f \circ f \circ f$ este strict crescatoare. (1p) Relatia din enunt implica

$f(f(f(x))) = f(f(x)) - f(x) = f(x) - x - f(x) = -x$ $(\forall)$ $x \in M$ contradictie. (2p)

b) $arcctg(ctgx) = x - np$, pentru $np < x < (n+1)p$, $n \in Z$ (1p)

$g(x) = p \cdot \left\{ \frac{1}{p} x \right\} = x - np$, (1p) deoarece $n < \frac{x}{p} < n+1 \Rightarrow \left\{ \frac{x}{p} \right\} = \frac{x}{p} - n$ (1p)

adica $g(x) = p\left(\frac{x}{p} - n \right) = x - np$. (1p)

Deci $A = R - \{kp \mid k \in Z\}$. (1p)

II. Din prima inegalitate, cu $x \to x + \frac{b}{a}$, obtinem: (2p)

$f(x) + 2\left(x + \frac{b}{a} \right) \le \frac{a}{b}\left(x + \frac{b}{a} \right)^2 + 2\frac{b}{a} \Leftrightarrow f(x) \le \frac{a}{b} x^2 + \frac{b}{a}$ (1) (2p)

Din a doua inegalitate, cu $x \to x - \frac{b}{a}$ se obtine: (2p)

$\frac{a}{b}\left(x - \frac{b}{a} \right)^2 + 2\frac{b}{a} \le f(x) - 2\left(x - \frac{b}{a} \right) \Leftrightarrow \frac{a}{b} x^2 + \frac{b}{a} \le f(x)$ (2) (2p)

Din (1) si (2) rezulta $f(x) = \frac{a}{b} x^2 + \frac{b}{a}$. (1p)

III. **1.** f si g sunt strict crescatoare deoarece $f = u + v$, $g = w + v$ (2p)

unde $u, v, w : R_+ \to R$ $u(x) = x^3$, $v(x) = x$, $w(x) = x^4$. (2p)

2. Avem $f\left(\frac{5}{4} \right) = \frac{205}{64} > 3 = f(a)$ implica $a < \frac{5}{4}$. (2p)

Din $g\left(\frac{5}{4} \right) = \frac{945}{256} < 4 = g(b)$ rezulta $b > \frac{5}{4}$. (2p)

Deci $a < b$ si $c > 0$. (1p)

IV. Avem: $\frac{2}{3}(m_a - m_b) = b - a \Leftrightarrow \frac{2}{3} \cdot \frac{3}{4}(b-a)(b+a) = (b-a)(m_a + m_b)$ (2p)

Cazul 1. Daca $a \neq b \neq c \neq a \Rightarrow$ (1p) $m_a + m_b = \frac{1}{2}(a+b)$ si analoagele. (1p)

Rezulta: $2(m_a + m_b + m_c) = a + b + c$, fals. (1p)

(deoarece $m_a > \dfrac{b+c-a}{2}$, etc.)

Cazul 2. Daca $b = c \neq a \Rightarrow$ (1p)

$$\begin{cases} m_a + m_b = \frac{1}{2}(a+b) \\ m_a - m_b = \frac{3}{2}(b-a) \end{cases} \Rightarrow \begin{cases} 2m_a = 2b - a \\ 2m_b = 2a - b \end{cases}$$ (1p)

$\Rightarrow 4m_a^2 = 4b^2 - 4ab + a^2 \Leftrightarrow$ (1p)

$2(b^2 + b^2) - a^2 = 4b^2 - 4ab + a^2 \Leftrightarrow 4ab = 2a^2 \Leftrightarrow a = 2b$, fals deoarece $b + c > a$. (1p)

1 punct din oficiu

Concursul de matematica Arhimede

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Solutii. Clasa a XI -a

I. $\begin{vmatrix} 56 & 64 \\ 63 & 72 \end{vmatrix} = 4032 - 4032 = 0$, (1p)

Deci X verifica ecuatia $\det(X^7) = 0$; cum $\det(X^7) = (\det X)^7 = 0$, rezulta $\det(X) = 0$. (1p)

Deci X este de forma $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ cu $a, b, c, d \in R$ si $ad - bc = 0$. (1p)

Din relatia lui Cayley $X^2 - (a+d)X + \det(X)I_2 = 0_2$, rezulta $X^2 - (a+d)X = 0_2$ sau $X^2 = (a+d)X$... Prin inductie gasim $X^n = (a+d)X^{n-1}$ oricare $2 \le n \in N$. (1p)

Atunci

$X^7 = (a+d)^6 X = \begin{pmatrix} (a+d)^6 a & (a+d)^6 b \\ (a+d)^6 c & (a+d)^6 d \end{pmatrix} = \begin{pmatrix} 56 & 64 \\ 63 & 72 \end{pmatrix}$ echivalenta cu sistemul $\begin{cases} (a+d)^6 a = 56 \\ (a+d)^6 b = 64 \\ (a+d)^6 c = 63 \\ (a+d)^6 d = 72 \end{cases}$ (1p)

Adunând prima ecuatie cu ultima gasim $(a+d)^7 = 128$ care în R da $a + d = 2$. (1p)

Atunci din prima ecuatie $a = \frac{56}{64} = \frac{7}{8}$, din a doua $b = \frac{64}{64} = 1$, din a treia $c = \frac{63}{64}$, iar din a patra $d = \frac{72}{64} = \frac{9}{8}$. (2p)

Asadar $X = \begin{pmatrix} \frac{7}{8} & 1 \\ \frac{63}{64} & \frac{9}{8} \end{pmatrix}$. (1p)

II. **1)** Observam ca $x^3 - x^2 + \frac{4}{27} = \left(x - \frac{2}{3}\right)^2 \left(x + \frac{1}{3}\right) \ge 0$ $(\forall)$ $x \in [0,1]$. (4p)

2) Deoarece $\sqrt{x} - \sqrt[3]{x} \le 0$ $(\forall)$ $x \in [0,1]$ rezulta $\sup A = 0$. (2p)

Deoarece $\left(\sqrt[6]{x}\right)^3 - \left(\sqrt[6]{x}\right)^2 + \frac{4}{27} \ge 0$ $(\forall)$ $x \in [0,1]$ rezulta $\inf A = -\frac{4}{27}$. (3p)

III. Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Notam $x_k = TrA^k \overset{(1p)}{\Rightarrow} x_{k+2} - (a+d)x_{k+1} + (ad - bc)x_k = 0$, $k \ge 0$. (2p)

Dar $x_n = x_{n+1} = 0 \Rightarrow x_{n+k} = 0$, $(\forall)$ $k \ge 0$. (2p)

Fie $B = A^n \Rightarrow trB = 0$ si $trB^2 = 0 \Rightarrow B^2 = 0_2$ si $A^{2n} = 0_2 \Rightarrow A^2 = 0_2$. (2p)

(Am tinut cont ca $A^k = 0 \Rightarrow \det A = 0 \Rightarrow A^2 = lA \Rightarrow A^k = l^{k-1}A = 0_2 \Rightarrow l = 0$ sau $A = 0_2$

$\Rightarrow A^2 = 0_2$). (2p)

IV. Sa notam $A = (x-y)(f(x) - f(y))$ (1p)

Avem:

$$A = (x-y)(f(x) - f(y)) = (x-y)\left(x - y + \frac{1}{x^2} - \frac{1}{y^2}\right) = (x-y)\left(x - y + \frac{y^2 - x^2}{x^2y^2}\right) = (x-y)^2\left(1 - \frac{x+y}{x^2y^2}\right) \text{(2p)}$$

Cazul I. $x, y \in \left(0, \sqrt[3]{2}\right)$ (1p)

Rezulta $\frac{x+y}{x^2y^2} = \frac{1}{xy^2} + \frac{1}{x^2y} \geq 1$. Deci $A \leq 0$ si prin urmare f este descrescatoare pe $\left(0, \sqrt[3]{2}\right)$. (2p)

Cazul II. $x, y \in \left[\sqrt[3]{2}, \infty\right)$ (1p)

Rezulta $\frac{x+y}{x^2y^2} = \frac{1}{xy^2} + \frac{1}{x^2y} \leq 1$. Deci $A \geq 0$ si prin urmare f este crescatoare pe $\left(\sqrt[3]{2}, \infty\right)$. (1p)

Deci $f(x) \geq f\left(\sqrt[3]{2}\right)$ $(\forall)$ $x \in (0, \infty)$. În concluzie $a = \sqrt[3]{2}$. (1p)

Concursul de matematica Arhimede

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Solutii. Clasa a XII-a

I. **1)** Aratam ca oricare $x, y \in G$ avem $x * y \in G$. Într-adevar din $x > 0$, $y > 0$, $a > 1$ rezulta $a^x > 1$, $a^y > 1$, de unde $\sqrt{a^x} - 1 > 0$, $\sqrt{a^y} - 1 > 0$. (1p)

De aici $\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^y} - 1\right) + 1 > 1$ (1p) si tinând seama de monotonia functiei logaritmice cu baza supraunitara, obtinem $2\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^x} + 1\right) + 1\right] > 0$, adica $x * y \in G$. (1p)

2) G_1. asociativitatea: (1,5p)

$$\left(x * y\right) * z = 2\log_2\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^y} - 1\right) + 1\right] * z = 2\log_a\left[\left(\sqrt{a^{2\log_a\left[\left(\sqrt{a^2} - 1\right)\left(\sqrt{a^y} - 1\right) + 1\right]}} - 1\right)\left(\sqrt{a^z - 1}\right) + 1\right] =$$

$$= 2\log_a\left\{\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^y} - 1\right) + 1 - 1\right]\left(\sqrt{a^z} - 1\right) + 1\right\} = 2\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^y} - 1\right)\left(\sqrt{a^z} - 1\right) + 1\right];$$ apoi

$$x * \left(y * z\right) = x * 2\log_a\left[\left(\sqrt{a^y} - 1\right)\left(\sqrt{a^z} - 1\right) + 1\right] = 2\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^{2\log_a\left[\left(\sqrt{a^y} - 1\right)\left(\sqrt{a^z} - 1\right) + 1\right]}} - 1\right) + 1\right] =$$

$$= 2\log_a\left\{\left(\sqrt{a^x} - 1\right)\left[\left(\sqrt{a^y} - 1\right)\left(\sqrt{a^z} - 1\right) + 1 - 1\right] + 1\right\} = 2\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^y} - 1\right)\left(\sqrt{a^z} - 1\right) + 1\right],$$ de unde rezulta asociativitatea.

G_4. comutativitatea: (1,5p)

$x * y = 2\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^y} - 1\right) + 1\right] = \log_a\left[\left(\sqrt{a^y} - 1\right)\left(\sqrt{a^x} - 1\right) + 1\right] = y * x$ oricare $x, y \in G$.

G_2. „*” admite element neutru: (1,5p) $(\exists)$ $u \in G$ a.î. $x * u = u * x = x$ oricare $x \in G$; Din motive de comutativitate, trebuie sa avem

$2\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^u} - 1\right) + 1 = x\right]$, adica

$\log_a\left[\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^u} - 1\right) + 1\right] = \dfrac{x}{2}$, sau

$\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^u} - 1\right) + 1 = a^{\frac{x}{2}}$ sau încât

$\left(\sqrt{a^x} - 1\right)\left(\sqrt{a^u} - 1\right) = \sqrt{a^x} - 1$ si cum $\sqrt{a^x} - 1 \neq 0$ oricare $x \in G$, rezulta $\sqrt{a^u} - 1 = 1$, de unde $\sqrt{a^u} = 2$, adica $a^n = 4$ si deci $u = \log_a 4 \in G$.

G_3. orice element din G este simetricabil: (1,5p)

$(\forall)$ $x \in G$ $(\exists) x' \in G$ a.î. $x * x' = x' * x = u$.

Din motiv de comutativitate

$2\log_a\left[\left(\sqrt{a^x}-1\right)\left(\sqrt{a^{x'}}-1\right)+1\right]=\log_a 4$ echivalenta cu $2\log_a\left[\left(\sqrt{a^x}-1\right)\left(\sqrt{a^{x'}}-1\right)+1\right]=2\log_a 2$, de unde $\left(\sqrt{a^x}-1\right)\left(\sqrt{a^{x'}}-1\right)+1=2$, adica $\left(\sqrt{a^x}-1\right)\left(\sqrt{a^{x'}}-1\right)=1$; obtinem $\sqrt{a^{x'}}-1=\frac{1}{\sqrt{a^x}-1}$, adica

$\sqrt{a^{x'}}=1+\frac{1}{\sqrt{a^x}-1}$ si deci $a^{x'}=\left(1+\frac{1}{\sqrt{a^x}-1}\right)^2$, de unde

$x'=\log_a\left(1+\frac{1}{\sqrt{a^x}-1}\right)^2=2\log_a\left(1+\frac{1}{\sqrt{a^x}-1}\right)>0$.

II. Avem $I=\int\frac{x\sin 2x+\cos 2x}{x^3}dx=\int\frac{2x\sin x\cos x+\cos 2x-1+1}{x^3}dx=$ (3p)

$=\int\frac{2x\sin x\cos x-2\sin^2 x}{x^3}dx+\int\frac{1}{x^3}dx\overset{(1p)}{=}\int\frac{2\sin x(x\cos x-\sin x)}{x^3}dx+\int\frac{dx}{x^3}=$ (1p)

$=2\int\frac{\sin x}{x}\cdot\frac{x\cos x-\sin x}{x^2}dx+\int\frac{dx}{x^3}\overset{(1p)}{=}2\int\frac{\sin x}{x}\cdot\left(\frac{\sin x}{x}\cdot\right)'dx+\int\frac{dx}{x^3}\overset{(2p)}{=}\frac{\sin^2 x}{x^2}-\frac{1}{2x^2}+C$. (1p)

III. **1)** Avem: $F\left(F^{-1}(x)\right)=x\ (\forall)\ x\in J$.

Deci $F'\left(F^{-1}(x)\right)\left(F^{-1}(x)\right)'(x)=1\ (\forall)\ x\in J$ (1p) si prin urmare $\left(F^{-1}\right)'(x)=\frac{1}{f\left(F^{-1}(x)\right)}$, $x\in J$. (1p)

$\left(\frac{1}{2}\left[F^{-1}(x)\right]^2\right)'=F^{-1}(x)\left(F^{-1}(x)\right)'=F^{-1}(x)\cdot\frac{1}{f\left(F^{-1}(x)\right)}$, $x\in J$. (2p)

Deci $\int\frac{F^{-1}(x)}{f\left(F^{-1}(x)\right)}dx=\frac{1}{2}\left[F^{-1}(x)\right]^2+C$, $x\in J$. (1p)

2) Sa consideram $f(x)=\frac{e^x+e^{-x}}{2}$, $x\in R$ si $F:R\to R$, $F(x)=\frac{e^x-e^{-e}}{2}$

Avem $F'=f$ si F bijectiva. (1p)

Din $F^{-1}(x)=In\left(x+\sqrt{x^2+1}\right)$, $x\in R$ rezulta $f\left(F^{-1}(x)\right)=\frac{e^{F^{-1}(x)}+e^{-F^{-1}(x)}}{2}=\sqrt{x^2+1}$, $x\in R$. (1p)

Deci $\frac{F^{-1}(x)}{f\left(F^{-1}(x)\right)}=\frac{In\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$, $x\in R$ (1p)

Rezulta $\int\frac{In\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}dx=\frac{1}{2}In^2\left(x+\sqrt{x^2+1}\right)+C$, $x\in R$. (1p)

IV. $A^k=\hat{3}^{k-1}A$, $(\forall)\ k\in N^*$. (1p)

1) $\Rightarrow$2), $(3,n)=1 \Rightarrow \hat{3}\in U_{Z_n}$ (multimea elementelor inverse) (1p)

$(U_{Z_n},\cdot)$ e grup finit $\Rightarrow$

$\dot{3}^{j(n)}=\dot{1}$ (Euler) (sau $\dot{3}^{ord\,\mathrm{gr}}=\dot{1}$) $\Rightarrow j(n)=k \Rightarrow A^k=I \Rightarrow$ (1p)

$G=\{A, A^2,...,A^k\}$ pentru ca $A^{k+1}=3^k\cdot A=A\cdot \Rightarrow$ (G,·) grup (1p)

2)$\Rightarrow$1) Daca, $(G,\cdot)$ e grup fie E element neutru.

$$\Rightarrow E=A^k \text{(pentru un k)} \quad \text{(1p)} \Rightarrow A^{k+1}=A \Rightarrow A^{k+1}=3^k A \overset{(1p)}{\Rightarrow} (\dot{3}^k-\dot{1})A=\begin{pmatrix}0..........0\\ ....\\ 0..........0\end{pmatrix}=0_3 \quad (2p)$$

$\Rightarrow \dot{3}^k-\dot{1}=\dot{0} \Rightarrow \dot{3}^k=\dot{1} \Rightarrow (n,3)=1$. (1p)