Barem de corectare

Clasa a VII-a

1. a)Demonstrati ca $\sqrt{35n^2+42n+10} \in$ R – Q, oricare ar fi n numar rational.

$35n^2+42n+10=7(5n^2+6n+1)+3$……………………………………………………….1p

a $\in \{7k, 7k+1, 7k+2, 7k+3, 7k+4, 7k+5, 7k+6\}$, daca a $\in$ N

$(7k)^2=49k^2=7p$

$$\left.\begin{array}{l}(7k)^2=49k^2=7p\\(7k+1)^2=49k^2+14k+1=7(7k^2+2k)+1=7p+1\\(7k+2)^2=\dots\dots\dots\dots\dots\dots=7p+4\\(7k+3)^2=\dots\dots\dots\dots\dots\dots=7p+2\\(7k+4)^2=\dots\dots\dots\dots\dots\dots=7p+2\\(7k+5)^2=\dots\dots\dots\dots\dots\dots=7p+4\\(7k+6)^2=\dots\dots\dots\dots\dots\dots=7p+1\end{array}\right\}\Rightarrow \quad 2p$$

$p \in N$

$a^2 \in \{7p, 7p+1, 7p+2, 7p+4 \mid p \in N\} \Rightarrow 7p+3 \neq a^2$

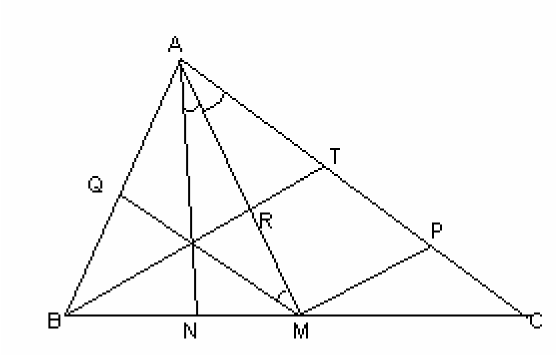
$\Rightarrow \sqrt{35n^2+42n+10}=\sqrt{7(5n^2+6n+1)+3} \in$ R-Q…………………………………………1p

b) Sa se arate ca numarul:

a= $\underbrace{22......2}_{n\_cifre}{}^4-8\cdot\underbrace{22...23}_{n\_cifre}{}^2+8\cdot\underbrace{44...47}_{n\_cifre}$ este patrat perfect, oricare ar fi n $\in$ N

$$\left.\begin{array}{l}22...23=22...22+1=k+1.............1p\\44...47=44...44+3=2\cdot 22...22+3=2k+3\\22....22=k......................................1p\end{array}\right\}\Rightarrow a=k^4+8(k+1)^2+8(2k+3)=(k^2-4)^2 \ \dots 1p$$

2. In triunghiul ABC, BC = 2AB, AM mediana ($M\in BC$), N mijlocul segmentului BM si $P\in AC$, astfel incat $MP \perp AM$. a) Demonstrati ca AM este bisectoarea unghiului $\hat{NAC}$. b) Calculati valoarea raportului $\frac{PC}{AP}$.



a) In ΔABM is. fie MQ mediana……………………………1p

Demonstram ca $\hat{MAN} \equiv \hat{AMQ}$ ………………………… 1p

MQ l.m. $\Rightarrow$ MQ || AC ……………………………………1p

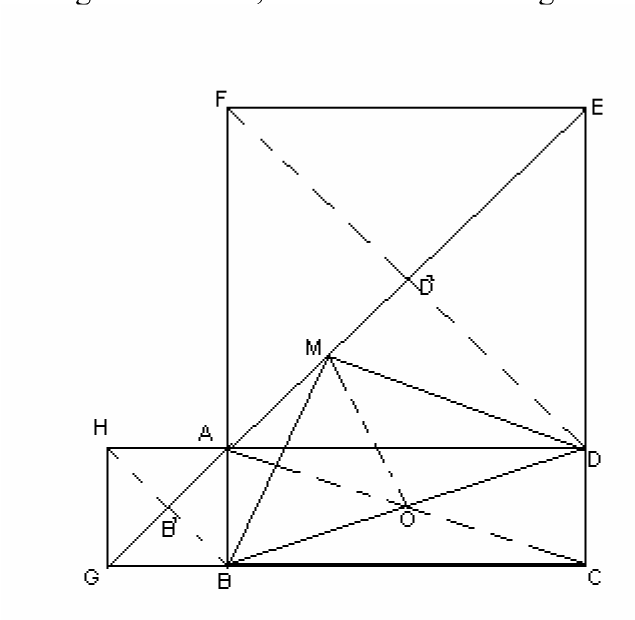
$\Rightarrow \hat{AMQ} \equiv \hat{MAC} \Rightarrow \hat{MAN} \equiv \hat{MAC}$ ………………………1p

b) ΔABM is. fie BR mediana $\Rightarrow \left.\begin{matrix} BR \perp AM \\ MP \perp AM \end{matrix}\right\} \Rightarrow BR \parallel MP$…………………………………………1p

BR $\cap$ AC={T}

$\left.\begin{matrix} \Delta AMP, RT l.m. \Rightarrow AT = TP.........1p \\ \Delta BTC, MP l.m. \Rightarrow TP = PC.........1p \end{matrix}\right\} \Rightarrow AT = TP = PC \Rightarrow \frac{PC}{AP} = \frac{1}{2}$……………………………… 1p

3. In exteriorul dreptunghiului ABCD se construiesc patratele ADEF, ABGH. Daca M este mijlocul segmentului GE, calculati masurile unghiurilor triunghiului MBD.



ΔGCE dr. is. $\Rightarrow CM \perp GE \Rightarrow \Delta AMC$ dr……1p

Se demonstreaza ca G,A,E coliniare…………..1p

Fie $AC \cap BD = \{O\} \Rightarrow OA=OB=OC=OD$……..1p

$\Rightarrow OM = \frac{AC}{2} = \frac{BD}{2} \Rightarrow \Delta MBD dr$...........1p

$$B'D' = \frac{GE}{2} \Rightarrow \left.\begin{array}{l} MD' = AB' = BB' ....1p \\ MB' = D'E = D'D ....1p \\ 9\hat{0} \end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta BB'M \equiv \Delta MDD \Rightarrow MD = MB \Rightarrow$

$\Rightarrow m\hat{MBD} = m\hat{MDB} = 45^\circ$ …………………..1p

Clasa a VIII a

1. a) Sa se determine multimile $D \subseteq R$ astfel incat functiile f,g : D$\rightarrow$R, f(x)=x^3+x^2+1 si g(x)=2x+1sa fie egale
   f(x)=g(x)$\Leftrightarrow$
   $x^3+x^2+1=2x+1 \Leftrightarrow x^3+x^2-2x=0 \Leftrightarrow x(x^2+x-2)=0 \Leftrightarrow x(x-1)(x+2)=0 \Leftrightarrow x \in \{-2,0,1\}$
   (1p) (1p)

D$\in$ {{-2,0,1},{-2,0},{-2,1},{0,1},{-2},{0},{1}}….(1p)

b) Fie n un numar natural impar. Demonstrati ca fractia $\frac{n^3+2n^{2+3n+4}}{n^2+2}$ este ireductibila.

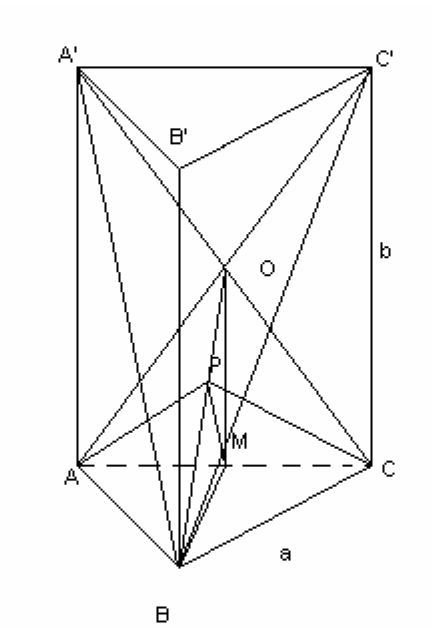
Fie (n^3+2n^2+3n+4, n^2+2)=d $\Rightarrow \left.\begin{array}{l} d/\ n^3+2n^2+3n+4 \\ d/\ n^2+2 \Rightarrow d/\ n^3+2n \end{array}\right\} \Rightarrow \begin{array}{l} d/\ 2n^2+n+4 \\ d/\ n^2+2 \Rightarrow d/\ 2n^2+4 \end{array} \Rightarrow$

$d/\ n \Rightarrow \begin{array}{l} d/\ n^2 \\ d/\ n^2+2 \end{array} \Rightarrow d/\ 2$ ..(2p)

=>d=1(1p)

$\left.\begin{array}{l} d/n^2+2 \\ n\_impar => n^2+2\_impar \end{array}\right\}$=>d impar (1p)

2. Se considera prisma triunghiulara regulata ABCA'B'C'. Sa se demonstreze ca planele (ABC') si (BCA') sunt perpendiculare, daca si numai daca AA'=AB$\frac{\sqrt{6}}{2}$.



Fie {O}=AC'$\bigcap A'C$ si OM $\perp$ AC (1p)

AC$\perp$(BOM) (1p)

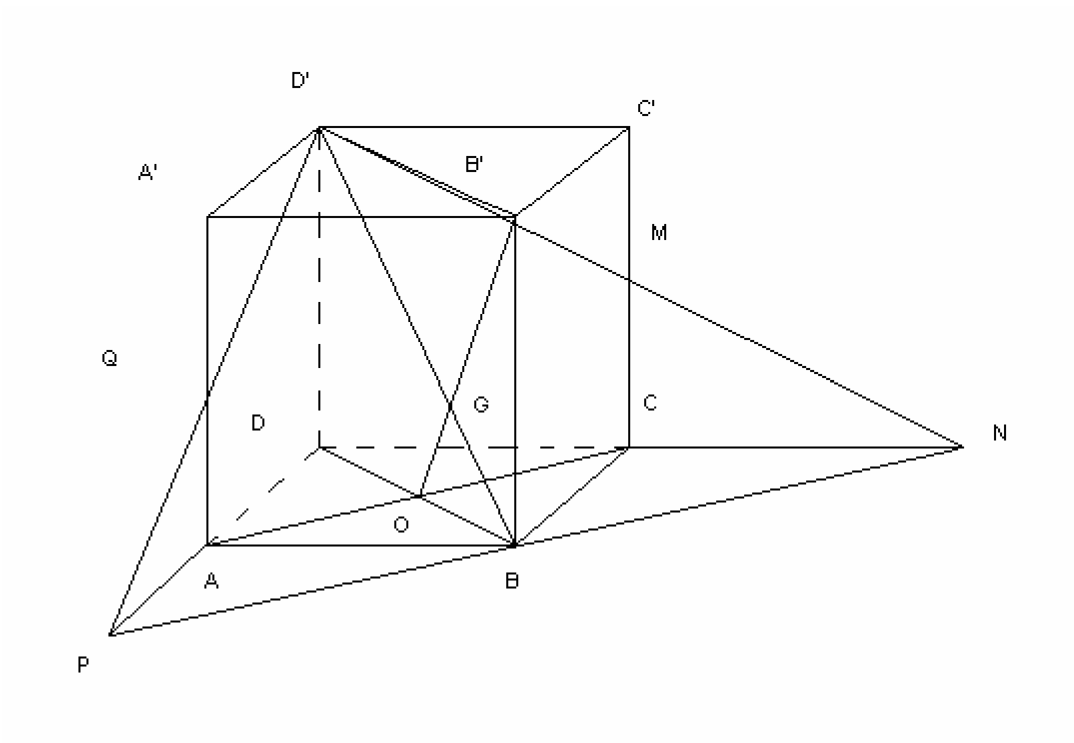
Fie MP$\perp$OB (1p)

$\widehat{(ABC'),(BCA')} = \widehat{APC}$ (1p)

$\widehat{APC} = 90°$ (1p)$\Leftrightarrow$ MP=$\frac{AC}{2} \Leftrightarrow \frac{OM * BM}{OB} = \frac{AC}{2}$ (1p) $\Leftrightarrow$

$\Leftrightarrow \frac{\frac{b}{2} * \frac{a\sqrt{3}}{2}}{\frac{\sqrt{b^2+3a^2}}{2}} = \frac{a}{2} <=> b = \frac{a\sqrt{6}}{2}$ (1p)

3. Fie cubul ABCD A'B'C'D' si M mijlocul medianei CC'. Planul (BMD') intersecteaza DC si DA in N respectiv P. a) Demonstrati ca triunghiurile D'NP si ACB' au acelasi centru de greutate. b) Stabiliti raportul volumelor determinate de planul (MBD') in cubul ABCDA'B'C'D' si studiati cum se modifica acest raport daca M∈CC' dar nu este mijlocul segmentului CC'.



a.Daca notam muchia cubului cu a

DN=2a, DP=2a….1p.

N,B,P coliniare, BP=BN⇒D'B mediana DD'PN…1p.

BD∩AC={o}, BD ∩B'D={G} B'O mediana triunghi B'AC ….1p.

triungiul OBG~triunghi B'D'G ⇒ $\frac{OG}{GB'} = \frac{BG}{GD'} = \frac{1}{2}$ ⇒ G centr de greutate coincid …..1p.

b. Fie D'P∩AA'={Q}⇒$V_{D'QABCMD} = V_{D'DPN} - V_{QAPB} - V_{MCBN} = \frac{a^3}{2}$ ⇒raportul=1….1p.

-daca M∈[CC'] demonstratie identica….1p.

-daca M∈CC' si M∉[CC'] planul taie alte doua muchii opuse ca la punctul anterior deci demonstratia nu se modifica…….1p.

Barem de corectare

Clasa a IX-a

1. Fie a,b,c $\in$ R astfel incat $|ax^2+bx+c|\le 1$,pentru orice x$\in[-1,1]$. Sa se arate ca $|2ax+b|\le 4$ pentru orice x$\in[-1,1]$.

Dan Radu, Bucuresti

Solutie :

Fie $f(x)=ax^2+bx+c,$

$f(0)=c, f(1)=a+b+c, f(-1)=a-b+c$

atunci : $2a=f(1)+f(-1)-2f(0), b=\frac{1}{2}(f(1)-f(-1)$

……….2p

Se obtine:

$$|2ax+b|=\left|(f(1)+f(-1)-2f(0))x+\frac{1}{2}(f(1)-f(-1))\right|=\left|f(1)(x+\frac{1}{2})+f(-1)(x-\frac{1}{2})-2xf(0)\right|\le$$

……….1p

$$\le|f(1)|\left|x+\frac{1}{2}\right|+|f(-1)|\left|x-\frac{1}{2}\right|+2|f(0)|\,|x|\le\left|x+\frac{1}{2}\right|+\left|x-\frac{1}{2}\right|+2|x|=g(x)$$

……….2p

$$g(x)=\begin{cases}-4x, x\in[-1,-\frac{1}{2}]\\ -2x+1, x\in(-\frac{1}{2},0]\\ 2x+1, x\in(0,\frac{1}{2})\\ 4x, x\in(\frac{1}{2},1]\end{cases}$$

Avem $2\le g(x)\le 4$ pentru $x\in[-\frac{1}{2},-1]\bigcup[\frac{1}{2},1]$ si $1\le g(x)\le 2$ pentru $x\in[-\frac{1}{2},\frac{1}{2}]$

Rezulta $|2ax+b|\le 4$ pentru orice $x\in[-1,1]$.

……….2p

2. Fie $(a_n)_{n\geq 0}$ progresie aritmetica de ratie pozitiva. Sa se rezolve in R ecuatia :

$$\sum_{i=1}^{2005}|x-a_{2i}|=\sum_{i=1}^{2005}|x-a_{2i+1}|$$

Paul Georgescu, Gabriel Popa, Iasi

Solutie:

Fie $r>0$ ratia progresiei. Ecuatia devine $\sum_{i=1}^{2005}(|x-a_{2i}|-|x-a_{2i+1}|)=0$

$$|x-2a_{2i}|-|x-a_{2i+1}|=\begin{cases}r, x\geq a_{2i+1}\\ 2x-(a_{2i}+a_{2i+1}, x\in[a_{2i},a_{2i+1}]\\ -r, x<a_{2i}\end{cases}$$

.........2p

Ecuatia nu are solutii pe $(-\infty,a_2]\bigcup[a_{2*2005+1},+\infty)$

.........1p

Pentru $x\in[a_{2k},a_{2k+1}), k=\overline{1,2005}$ se obtine

$\underbrace{-r-r-...-r}_{(k-1)ori}+(2x-(a_{2k},a_{2k+1}))+r+r+...+r=0$ sau $x=a_1+\dfrac{r(6k-2007)}{2}$

Se impune conditia $x\in[a_{2k},a_{2k+1})$ de unde $k=1003$ si $x=a_1+\dfrac{4011r}{2}=a_{2006}+\dfrac{r}{2}$

..........2p

Pentru $x\in[a_{2k-1},a_{2k}), k=\overline{1,2005}$ se obtine

$\underbrace{-r-r-...-r}_{k\_ori}+\underbrace{r+r+...+r}_{2005-k}=0 <=> r(2005-2k)=0$ imposibil

Ramane ca ecuatia are solutie unica $x=a_{2006}+\dfrac{r}{2}$

..........2p

3. Se considera un triunghi ABC si un numar real $k > 1$.Pe laturile BC,CA,AB se aleg punctele D,E,F astfel incat $\frac{BC}{BD} = \frac{CA}{CE} = \frac{AB}{AF} = k$. Se noteaza cu G,H,I intersectiile dreptelor BE si AD,CF si BE,respectiv AD si CF. Sa se discute in functie de k daca exista punctele X,Y,Z pe laturile BC,CA,AB astfel incat H$\in$(XY), I$\in$(YZ) si G$\in$(ZX).

Dan Branzei, Iasi

Solutie :

Proiectand triunghiul pe un anumit plan, se poate obtine un triunghi echilateral, iar rapoartele se mentin prin proiectii. Admitem ca triunghiul ABC este echilateral, de latura 1.

……….3p

Atunci triunghil GHI este echilateral si triunghiul XYZ poate fi cautat, de asemenea, echilateral.

……….1p

Consideram punctele X,Y,Z astfel incat :BX=CY=AZ=$t \in (0,1)$.

Din teorema lui Menelaos se obtine : $\frac{BH}{HE} = k-1$ si $\frac{YE}{YC} = \frac{HE}{HB} * \frac{XB}{XC}$ de unde

$kt = \frac{(k-1)(1-t)}{(k-1)-kt}$ sau $k^2t^2 - (k^2-1)t + k - 1 = 0$

………..2p

Pentru ca punctele X,Y,Z sa existe, avand proprietatea din enunt, trebuie ca $\Delta \geq 0$ sau $k^3 - 3k^2 - k - 1 \geq 0$.

………..1p

CLASA A –X-A

1. Sa se rezolve in R ecuatia:

$\log_2(\sin x)+\log_3(\text{tg } x)=\log_4(\cos^2 x)+\log_5(\text{ctg } x)$

Dorin Marghidanu,Corabia
(G.M. 2/2005)

<u>Solutie:</u>
Este suficient da determinam solutiile din $[0,2\pi)$.Din conditiile de existenta ale logaritmilor ,$x\in(0,\frac{\pi}{2})$…………..(1 punct)

Ecuatia devine: $\log_2(\sin x)+\log_3(\text{tg } x)=\log_2(\cos x)-\log_5(\text{tg } x)$ $\Leftrightarrow$
$\log_2(\text{tg } x)+\log_3(\text{tg } x)+\log_5(\text{tg } x)=0$ …………..(2 puncte)

Consideram $f:(0,\frac{\pi}{2})\to R$, $f(x)=\log_2(\text{tg } x)+\log_3(\text{tg } x)+\log_5(\text{tg } x)$, care este strict crescatoare

$f(\frac{\pi}{4})^3=0$; rezulta ca $x_0=\frac{\pi}{4}$ este unica solutie in $[0,2\pi)$ a ecuatiei…….(3 puncte)

Multimea solutiilor este $\{\frac{\pi}{4}+2k\pi/\ k\in Z\}$……….(1 punct)

2. Fie $z_1, z_2, z_3, z_4 \in C$ distincte doua cate doua ,de modul 1.Daca exista $a\in R \setminus\{-1\}$ astfel incat:

$$|az_1 + z_2 + z_3 + z_4|^2 + |z_1 + az_2 + z_3 + z_4|^2 + |z_1 + z_2 + az_3 + z_4|^2 + |z_1 + z_2 + z_3 + az_4|^2 = 4(a-1)^2$$

unde z_1,z_2,z_3,z_4 sunt afixele varfurilor unui dreptunghi.

Marian Ursarescu,Roman

Solutie:

Fie $z=z_1+z_2+z_3+z_4$;avem $\Sigma\ |z+(a-1)z_K|^2 = 4(a-1)^2 \Leftrightarrow$

$$4z\overline{z} + (a-1)z(\overline{z_1} + \overline{z_2} + \overline{z_3} + \overline{z_4}) + (a-1)\overline{z}(z_1 + z_2 + z_3 + z_4) + (a-1)(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) = 4(a-1)$$

$$\Leftrightarrow\ 4z\overline{z} + z(a-1)z\overline{z} = 0 \ \Leftrightarrow\ |z|^2(4+2a-2) = 0 \ \Leftrightarrow |z|^2(\underbrace{2a+2}_{\neq 0}) = 0 \Leftrightarrow |z| = 0 \Leftrightarrow z=0$$

(4puncte)

Daca $z_1+z_2+z_3+z_4=0$,atunci z_1,z_2,z_3,z_4 sunt varfurile unui paralelogram….

(2 puncte)

Un paralelogram inscriptibil este dreptunghi………………. (1 punct)

3. Fie $P_1, P_2, \ldots P_{13}$ puncte in plan ,oricare trei necoliniare si toate avand ambele coordonate intregi.Sa se arate ca exista cel putin un triunghi $P_iP_jP_k$ astfel incat centrul sau de greutate sa aiba ambele coordonate intregi.

Vasile Pravat & Titu Zvonaru ,Comanesti
Rec .Mat. 1/2005

Solutie :

Cel putin 5 abscise x_i dau acelasi rest la impartirea prin 3 (principiul cutiei),deci abscisa centrului de greutate este numar intreg,oricum am alege trei indici din multimea A a celor 5 determinanti astfel…. (3 puncte)

Fie M multimea resturilor modulo 3 ale numerelor $\{y_i \mid i \in A\}$.Daca M are 3 elemente ,alegem $i,j,k \in A$ astfel incat $y_1+y_2+y_3 \equiv 0+1+2 \pmod 3$ si gata…(2 puncte)

Daca M are 2 elemente, cel putin 3 ordonate dau acelasi rest la impartirea prin 3 si le alegem pe ele……….. (1 punct)

Daca M are un singur element,concluzia este imediata………………(1 punct).

BAREM CORECTARE CLASA A XI A

1. Fie A, B, $A^{-1} \in M_2(Z)$ si multimea $M=\{k\in Z/ (A+KB)^{-1}\in M_2(Z)\}$. Se se arate ca cardinalul $M\neq 2006$.

Prof. Gabriel Constantin, Roman

Solutie :

Pt. X, $x^{-1}\in M_2(Z) \Rightarrow \det x=\pm 1 \Rightarrow \det(A+KB)=\pm 1$ 2p.

$\det(A+KB)=K^2\det B + \alpha K + \det A= f(k)$

daca $\exists\ k_1, k_2, k_3, k_4$ a.i. $(A+K_i B)^{-1}\in M_2(Z)$, $i=\overline{1,4}$ atunci $f(k_1), f(k_2), f(k_3), f(k_4), f(0)\in\{\pm 1\} \Rightarrow f$ ia valoarea 1 sau –1 pentru 3 valori distincte$\Rightarrow$f= constant 3p.$\Rightarrow\exists$ o infinitate de valori K pentru care $(A+KB)^{-1}\in M_2(Z) \Rightarrow$cardinalul $M\neq 2006$ 2p.

Nota ! Pentru demonstatie pentru A si B particulare. 2p.

2. Fie $A \in M_3(R)$ inversabila a.i. $trA = trA^2 = 0$.
Sa se arate ca $\det(A+A^{-1}) = \det A + \det A^{-1}$

Prof. Marian Ursarescu, Roman

Solutie : $\det(A+A^{-1}) = \det(A^{-1}(A^2+I_3)) = \det A^{-1} \det(A^2+I_3) = \frac{\det(A^2+I_3)}{\det A}$ (1)

Fie $p_a(x) = x^3 - trA\, x^2 + trA^A\, x - \det A$

$trA = 0$ $trA^A = \frac{1}{2}((tr\, A)^2 - tr\, A^2) = 0 \Rightarrow p_A(x) = x^3 - \det A$ au valorile proprii $\lambda_1, \lambda_2, \lambda_3$. Fie $f(x) = x^2 + 1$. 1p.

$\det(A^2+I_3) = \det f(A) = f(\lambda_1)\, f(\lambda_2)\, f(\lambda_3) = (\lambda_1^2+1)\,(\lambda_2^2+1)\,(\lambda_3^2+1)$ 1p.

$p_A(x) = (x-\lambda_1)\,(x-\lambda_2)\,(x-\lambda_3)$

$p_i(x) = (i-\lambda_1)\,(i-\lambda_2)\,(i-\lambda_3)$

$p_{-i}(x) = (-i-\lambda_1)\,(-i-\lambda_2)\,(-i-\lambda_3)$ 2p.

$p_A(i)\, p_A(-i) = (i^2-\lambda_1^2)\,(i^2-\lambda_2^2)\,(i^2-\lambda_3^2) = (1+\lambda_1^2)\,(1+\lambda_2^2)\,(1+\lambda_3^2)$

$\det(A^2+I_3) = (-i - \det A)\,(i - \det A) = -(i + \det A)\,(i - \det A) = -(i^2 - (\det A)^2) = 1 + (\det A)^2$ (2) 1p.

Din (1) si (2) $\Rightarrow \det(A+A^{-1}) = \frac{1+(\det A)^2}{\det A} = \det A + \frac{1}{\det A} \Rightarrow \det(A+A^{-1}) = \det A + \det A^{-1}$ 1p.

3. Fie sirurile $(a_n)_{n\geq 0,}$ $(b_n)_{n\geq 0,}$ $(c_n)_{n\geq 0,}$ definita astfel: a_0, b_0, $c_0 >0$

si $\sqrt{a_{n+1}+a_{n+1}b_n}+\sqrt{a_{n+1}+a_{n+1}c_n}=\sqrt{b_n+b_nc_n}+\sqrt{a_{n+1}+a_{n+1}b_n}$

$\sqrt{b_{n+1}+b_{n+1}c_n}+\sqrt{b_{n+1}+b_{n+1}a_n}=\sqrt{c_n+c_na_n}+\sqrt{a_n+a_nc_n}$

$\sqrt{c_{n+1}+c_{n+1}a_n}+\sqrt{c_{n+1}+c_{n+1}b_n}=\sqrt{a_n+a_nb_n}+\sqrt{b_n+b_na_n}$

Sa se arate ca :

$\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n=\lim_{n\to\infty} c_n=tg^2\ \dfrac{arctg\sqrt{a_0}+arctg\sqrt{b_0}+arctg\sqrt{c_0}}{3}$

prof. Gabriel Necula, Plopeni

Solutie: Prin inductie dupa $n\geq 1$, din conditia de existenta asupra primului radical, in fiecare relatie $\Rightarrow a_n\geq 0$, $b_n\geq 0$, $c_n\geq 0$, $\forall n\in N^*$. Pentru orice a_n, b_n, $c_n\geq 0$ exista x_n, y_n, $z_n\in[o,\ \frac{\pi}{2})$ a.i.

$tg\ x_n=\sqrt{a_n}$, $tg\ y_n=\sqrt{b_n}$, $tg\ z_n=\sqrt{c_n}$ 3p.

Din prima relatie se obtine

$\sqrt{tg^2x_{n+1}+tg^2x_{n+1}tg^2y_n}+\sqrt{tg^2x_{n+1}+tg^2x_{n+1}tg^2z_n}=\sqrt{tg^2y_n+tg^2y_ntg^2z_n}+\sqrt{tg^2z_n+tg^2z_ntg^2y_n}\Rightarrow$

$\Rightarrow tg\ x_{n+1}(\frac{1}{\cos y_n}+\frac{1}{\cos z_n})=tg\ z_n\ \frac{1}{\cos z_n}+tg\ z_n\ \frac{1}{\cos y_n}$ de unde $tg\ x_{n+1}=tg\ \frac{y_n+z_n}{2}$.

Analog $tg\ y_{n+1}=tg\ \frac{z_n+x_n}{2}$ si $tg\ z_{n+1}=tg\ \frac{y_n+z_n}{2}$.

Rezulta $x_{n+1}=\frac{y_n+z_n}{2}$, $y_{n+1}=\frac{z_n+x_n}{2}$, $z_{n+1}=\frac{y_n+z_n}{2}$ (1)

Adunand $x_{n+1}+y_{n+1}+z_{n+1}=x_n+y_n+z_n$ $\forall n\in N$ de unde

$x_n+y_n+z_n=x_0+y_0+z_0=\alpha\in R$, $\forall n\in N$ 2p.

Inlocuind in (1) se obtin relatiile :

$x_{n+1}=\frac{\alpha}{2}-\frac{1}{2}x_n$

$y_{n+1}=\frac{\alpha}{2}-\frac{1}{2}y_n$ $\forall n\in N$

$z_{n+1}=\frac{\alpha}{2}-\frac{1}{2}z_n$

Pentru sirul x_n se obtine $x_n=\frac{\alpha}{2}[1-(-\frac{1}{2})^n]+(-\frac{1}{2})^n x_0$ 1p.

Atunci $\lim_{n\to\infty} a_n=\lim_{n\to\infty} tg^2\ x_n=tg^2\frac{x_0+y_0+z_0}{3}\Rightarrow\lim_{n\to\infty} a_n=tg^2\ \dfrac{arctg\sqrt{a_0}+arctg\sqrt{b_0}+arctg\sqrt{c_0}}{3}$

Analog pentru y_n si z_n. 1p.

BAREM DE CORECTARE

CLS A XII-A

1. Fie I(a) = $\int_0^1 \frac{arctgx}{x^2+x+a}$ dx , a >o.

   a) Sa se arate ca I(1) ≤ $\frac{\pi}{3\sqrt{3}}(\frac{\pi}{4}-\ln\sqrt{2})$.

   b) Sa se calculeze I(2).

Prof. Marian Ursarescu

Dem.

a) f (x) = arctgx strict crescator ; g(x)=$\frac{1}{x^2+x+1}$ strcit descrectator; din inegalitatea Cebişev

$$\Rightarrow \int_0^1 \frac{arctgx}{x^2+x+1}dx \le \int_0^1 arctgx dx \cdot \int_0^1 \frac{1}{x^2+x+1}dx \qquad (1p)$$

$$\int_0^1 arctgx dx = \frac{\pi}{4}-\ln\sqrt{2} \qquad \int_0^1 \frac{1}{x^2+x+1}dx = \frac{\pi}{3\sqrt{3}} \qquad (1p)$$

$$\Rightarrow I(1) \le \frac{\pi}{3\sqrt{3}}\left(\frac{\pi}{4}-\ln\sqrt{2}\right) \qquad (1p)$$

b) I(2)=$\int_0^1 \frac{arctgx}{x^2+x+2}dx$ x=$\frac{1-t}{1+t} \Rightarrow dx = \frac{-2}{(1+t)^2}dt$ x=0 $\Rightarrow$t=0. (1p)

$$I(2)=\int_1^0 \frac{arctg\ \frac{1-}{1+x}}{\frac{(1-t)^2}{(1+t)^2}+\frac{1-t}{1+t}+2}\left(-\frac{2}{1+t^2}\right)dt = \int_0^1 \frac{arctg\ \frac{1-x}{1+x}}{t^2+t+2}dt \qquad (1p).$$

Dar arctg$\frac{1-x}{1+x}+arctgx=\frac{\pi}{4}$, ∀ x>-1 (1p)

$$I(2)=\int_0^1 \frac{\frac{\pi}{4}-arctgt}{t^2+t+2}dt = \frac{\pi}{4}\int_0^1 \frac{1}{t^2+t+2}dt - I(2)$$

$$\Rightarrow I(2) = \frac{\pi}{8}\int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{t}}{2}\right)^2}dt = \frac{\pi}{8}\cdot\frac{2}{\sqrt{7}}\cdot arctg\frac{t+\frac{1}{2}}{\frac{\sqrt{t}}{2}}\Big|_0^1 \qquad (1p).$$

Prof. Marian Ursarescu, Roman.

2. Fie f : R $\to R$ periodica , marginaita si astfel incat exista $x_0 \in$R pentru care $l_s(x_0)$, $l_d(x_0)$ exista , sunt finite si distrincte. Determinati a$\in$R penrtu care nu exista $\lim_{x\to\infty}\int_0^x (f(t)+a)dt$.

prof. Paul Georgescu si prof. Gabriel Popa, Iasi.

Solutie : Fie T$\in R_T^*$ o perioada a lui $\int$ si notam $F(x)=\int_0^x (f(t)+a)dt$, atunci

$F(x+(n+1)T)-F(x+nT)=\int_{x+nT}^{x+(n+1)T}(f(t)+a)dt=\int_0^T f(t)dt+aT \Rightarrow F(x+nT)=F(x)+n\left(\int_0^T f(t)dt+aT\right)$ $\forall$ n$\in N^*$, $\forall$ x$\in$[0,T) (1p). Cum f este marginita, rezulta ca F este marginita pe [0,T], iar dacva $\int_0^T f(t)dt+aT\neq 0$, rezulta imediat ca $\lim_{x\to\infty}F(x)=(+\infty)\cdot \operatorname{sgn}\left(\int_0^T f(t)dt+aT\right)$. (1p)

Fie acum $a=-\frac{1}{T}\cdot\int_0^T f(t)dt$ atunci $F(x+T)=\int_0^{x+t}(f(t)+a)dt=F(x)$, $\forall$ x$\in$R (1p) , deci F este periodica de perioada T. Presupunem prin absurd ca F este constanta . Atunci $F(x_1)=F(x_2)$, $\forall$ $x_1,x_2\in$R, deci $\int_{x_1}^{x_2}(f(t)+a)dt=0$, $\forall$ x1,$x_2\in$R. Pentru $\varepsilon>0$ fixat, putem gasi $\delta>0$ astfel ca $(l_s-\delta+a)(x_2-x_1)\le\int_{x_1}^{x_2}(f(t)+a)dt\le(l_s+\varepsilon+a)\cdot(x_2-x_1)$ pentru $x_0-\delta\le x_1<x_2<x_0$, deci $l_s-\varepsilon+a\le 0\le l_s+\varepsilon+a$. Cum ε era arbitrar , obtinem ca $l_s(x_0)$ +a =0 (2p). Analog , $l_d(x_0)$ +a=0, deci $l_s(x_0)=l_d(x_0)$= -a , contradictie cu ipoteza (1p). Ramaneca F este periodica si neconstanta si deci nu are limita la $+\infty$ in cazul in care $\int_0^T f(t)dt+aT\neq 0$ (1p).

3. Sa se rezolve in Z_{21} ecuatia : (1)$x^{12}+x^7+\hat{7}x^6+\widehat{14}x^4+\widehat{14}x^3+\hat{7}x=\hat{0}$.

prof. Gabriel Popa , Iasi.

Solutie : Evident ca $x_1=\hat{0}$ este solutie a ecuatiei . Cautam in continuare $\alpha\in\{1,2,3,\ldots,20]$ a.i $\hat{\alpha}\in Z_{21}$, sa fie solutie a ecuatiei. Distingem situatiile :
(0,5p)

(i) $(\alpha,21)=1$. Conform teoremei lui Euler , avem atunci ca $\alpha^{\varphi(21)}\equiv 1(\mathrm{mod}\,21)$ Cum $\varphi(21)=\varphi(3)\cdot\varphi(7)=2\cdot 6=12$, iar $\hat{\alpha}$ este solutie a ecuatiei (1), obtinem in Z_{21} egalitatea : $\hat{\alpha}^7+\hat{7}\hat{\alpha}^6+\widehat{14}\hat{\alpha}^4+\widehat{14}\hat{\alpha}^3+\hat{7}\hat{\alpha}+\hat{1}=\hat{0}\Leftrightarrow\left(\hat{\alpha}+\hat{1}\right)^7=\hat{0}\Leftrightarrow(3)\Leftrightarrow\hat{\alpha}+\hat{1}=\hat{0}\Leftrightarrow\hat{\alpha}=\widehat{20}$. Echivalenta (2) provine din formula binomului Newton, precum si din urmatoarele observatii : $\hat{C}_7^2=\widehat{21}=\hat{0}$; $\hat{C}_7^3=\widehat{35}=14$

Pentru (3) vom dovedi ca : $\hat{u}^7=\hat{0}\Leftrightarrow\hat{u}=\hat{0}$

Implicatia $''\Leftarrow''$ este evidenta. Pentru $''\Rightarrow''$, observam ca in ipoteza $\hat{u}^7=\hat{0}$, $\hat{u}$ nu poate fi nici inversabil in Z_{21}, de asemenea u nu poate fi nici par(in Z) ; ramane ca u $\in\{0,3,7,9,15\}$. Insa $\hat{3}^7=\hat{3},\hat{9}^7=\hat{9},\hat{7}^7=\hat{7},\widehat{15}^7=\widehat{15}$, dupa cum se poate verifica imediat. Deci $\hat{u}=\hat{0}$ (2,5p)

(ii) $\alpha\in\{7,14\}$. Cum $\hat{\alpha}$ solutie pentru (1), avem :

$\hat{\alpha}^{12}+\hat{\alpha}^7+\hat{7}\hat{\alpha}^6+\widehat{14}\hat{\alpha}^4+\widehat{14}\hat{\alpha}^3+\hat{7}\hat{\alpha}=\hat{0}\Leftrightarrow\hat{\alpha}^{12}+\hat{\alpha}^7+\hat{7}\left(\hat{\alpha}^6-\hat{\alpha}^4\right)+\hat{7}\left(\hat{\alpha}^3-\hat{\alpha}\right)=\hat{0}\Leftrightarrow$

$\hat{\alpha}^{12}+\hat{\alpha}^7+\hat{7}\hat{\alpha}^3\left(\hat{\alpha}^3+\hat{\alpha}\right)+\hat{7}(\hat{\alpha}^3-\hat{\alpha})=\hat{0}\Leftrightarrow(4)\Leftrightarrow\hat{\alpha}^{12}+\hat{\alpha}^7=\hat{0}\Leftrightarrow\hat{\alpha}^7\left(\hat{\alpha}^5+\hat{1}\right)=\hat{0}\Leftrightarrow(5)\Leftrightarrow$ Deoar

$\alpha^5+1\in M_3\Leftrightarrow\alpha=M_3-1\Leftrightarrow(ii)\Leftrightarrow\hat{\alpha}=\widehat{14}$

ece $3|\ \hat{\alpha}-\alpha$, $\forall\alpha\in Z$, atunci avem ca $\hat{7}\left(\hat{\alpha}^3-\hat{\alpha}\right)=\hat{0}$, ceea ce justifica (4). In plus , fiindca ipoteza (ii) arata ca $\alpha\in M_7$, avem imediat (5).

(iii) $\alpha\in\{\hat{3},\hat{6},\hat{9},\widehat{12},\widehat{15},\widehat{18}\}$ Cum $\hat{\alpha}$ solutie , avem:

Observam ca : a) $\alpha=M_7+1\Rightarrow\alpha^5+1=(M_7+1)^5+1=M_7+2$

b) $\alpha=M_7-1\Rightarrow\alpha^5+1=(M_7-1)^5+1=M_7$

c) $\alpha=M_7+1\Rightarrow\alpha^5+1=(M_7+2)^5+1=M_7+5$

d) $\alpha=M_7-2\Rightarrow\alpha^5+1=(M_7-2)^5+1=M_7-2$

e) $\alpha=M_7+3\Rightarrow\alpha^5+1=(M_7+3)^5+1=M_7+6$

f) $\alpha=M_7-3\Rightarrow\alpha^5+1=(M_7-3)^5+1=M_7-1$

rezulta ca in mod necesar $\alpha=M_7-1$, deci $\alpha=6$. pentru $\hat{\alpha}=\hat{6}$, se vede ca: $\hat{\alpha}^7\left(\hat{\alpha}^5+\hat{1}\right)=\hat{\alpha}^7\left(\hat{\alpha}+\hat{1}\right)\left(\hat{\alpha}^4-\hat{\alpha}^3+\hat{\alpha}^2-\hat{\alpha}+\hat{1}\right)=\hat{0}\left(in Z_{21}\right)$ (2p)

asadar solutiile ecuatiei sunt:

$x_1=\hat{0},x_2=\hat{6},x_3=\widehat{14},x_4=\widehat{20}$