L294. Să se arate că, pentru orice numere pozitive a, b, c, are loc inegalitatea: $a^3+b^3+c^3-\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \geq \frac{1}{2}|(a-b)(b-c)(c-a)|$.

Marian Tetiva, Bârlad

L295. Determinaţi valoarea minimă a numărului real pozitiv k, încât pentru orice numere reale pozitive a, b, c să aibă loc inegalitatea:

$$\sum \frac{a^2+b^2}{a^2+ab+b^2} + k\frac{ab+bc+ca}{a^2+b^2+c^2} \leq k+2.$$

Florin Stănescu, Găeşti

Training problems for mathematical contests

A. Junior highscool level

G286. The sets $(A_n)_{n\geq 0}$ are defined as follows: $A_0 = \{1, 2, \ldots, 2015\}$; A_1 is obtained after replacing each element of A_0 by the sum of all the other elements of A_0; similarly, A_2 is obtained by replacing each element of A_1 by the sum of all the other elements of A_1 and so on. If $n \in \mathbb{N}^*$, show that no elements being congruent modulo 2016 exist in A_n

Vlad Tuchiluş, pupil, Iaşi

G287. We have an infinite number of counters in eight colours. Establish which is the least number of counters that have to be arranged in a row so that, for any couple of distinct colours, two neighbor counters having these two colours to be found in the row.

Ioan Viorel Codreanu, Satulung, Maramureş

G288. Let a, b, c be natural numbers such that $3ab = 2c^2$. Show that the number $a^3+b^3+c^3$ is a composite number.

Lucian Tuţescu, Craiova and Marian Voinea, Bucureşti

G289. Let a, b, c, x, y, z be positive real numbers such that $a+b+c=3$. Show that

$$36\left(\frac{x^3}{(a+b)^2}+\frac{y^3}{(b+c)^2}+\frac{z^3}{(c+a)^2}\right) \geq (x+y+z)^3.$$

Robert Antohi, pupil, Iaşi

G290. Let x, y, z, n be real numbers with $x, y, z \geq 1$ and $n > 1$, such that $x^2+y^2+z^2 = n^2+2$. Show that $x+y+z \geq n+2$.

Titu Zvonaru, Comăneşti and Bogdan Ioniţă, Bucureşti

G291. If a, b, c are positive real numbers, show that

$$\frac{a}{2b+c}+\frac{b}{2c+a}+\frac{c}{2a+b}+\frac{ab+bc+ca}{3(a^2+b^2+c^2)} \geq \frac{4}{3}.$$

Mircea Lascu and Marius Stănean, Zalău

G292. An acute-angle triangle ABC has the property that the sum of distances from any interior point of the triangle to the three sides equals the length of the altitude from A. Prove that the triangle is equilateral.

Petru Asaftei, Iaşi

G293. The circles $\mathcal{C}_1(O_1, r_1)$ and $\mathcal{C}_2(O_2, r_2)$ pass through the point M and cut the circles, for the second time, at the points A_1 and A_2, (respectively) at B_1 and B_2. If $MA_1 = A_1A_2$ and $MB_1 = B_1B_2$, show that the circles are tangent at M and $r_2 = 2r_1$.

Romanţa Ghiţă and Ioan Ghiţă, Blaj

G294. The points M and respectively D are considered on the sides BC and AC of the triangle ABC, and let $\{F\} = BD \cap AM$. The parallel through F to BC cuts AC at the point E. If $\dfrac{1}{DE} = \dfrac{1}{AD} + \dfrac{1}{CE} + \dfrac{1}{AC}$, show that M is the midpoint of BC.

Titu Zvonaru, Comăneşti

G295. Let us consider the triangle ABC and the points $A_1 \in BC$, $B_1 \in AC$ and $C_1 \in AB$ so that the lines AA_1, BB_1 and CC_1 be concurrent. The points X, Y and Z represent the intersections of the lines AP, BP and CP with the segments (B_1C_1), (A_1C_1) and respectively (A_1B_1), where P is a fixed point in the interior of the triangle ABC. Show that the lines A_1X, B_1Y and C_1Z are concurrent.

Neculai Roman, Mirceşti, Iaşi

B. Highschool level

L286. Let ABC be a triangle with its largest angle $\widehat{A}$ and the points $D_1, D_2 \in (BC)$ such that $\widehat{BAD_1} \equiv \widehat{ACB}$ and $\widehat{CAD_2} \equiv \widehat{ABC}$. If r is the radius of the incircle of $\triangle ABC$ and ρ is the radius of the circumcircle of $\triangle AI_1I_2$ (where I_1, I_2 are the centres of the circles inscribed in the triangles ABD_2, respectively ACD_1), prove that $r = 2\rho \sin^2 \dfrac{A}{2}$.

Neculai Roman, Mirceşti, Iaşi

L287. Let A', B', C' be the midpoints of the sides of the triangle ABC. Let S' denote the Spieker point of the triangle $A'B'C'$. The lines AS', BS', CS' cut the sides BC, CA, AB at the points M, N, and respectively P. Prove that the perpendicular lines at the points M, N, P on the sides BC, CA and respectively AB are concurrent if and only if the triangle is isosceles.

Nela Ciceu, Bacău and Roxana Mihaela Stanciu, Buzău

L288. If, in a non-isosceles triangle, the straight line determined by Lemoine′s point and the centre of Euler′s circle is parallel to a side of the triangle, then the triangle is right-angled.

Titu Zvonaru, Comăneşti and Neculai Stanciu, Buzău

L289. We build the sequence of triangles $A_nB_nC_n$, $n \in \mathbb{N}$, as follows: $\triangle A_0B_0C_0$ is arbitrarily chosen; the vertices of the triangle $A_{k+1}B_{k+1}C_{k+1}$ are the points at which the medians of $\triangle A_kB_kC_k$ intersect the circumcircle of this triangle, for any $k \in \mathbb{N}$. If two congruent triangles exist in the sequence defined in this way, show that

$\triangle A_0B_0C_0$ is equilateral.

Vasile Jiglău, Arad

L290. Let $ABCD$ be an orthodiagonal quadrilateral which is both inscribable and circumscribable. Let r, R be the radii of the inscribed and respectively circumscribed circles. If $a = AB$, $b = BC$, $c = CD$, $d = DA$, $e = AC$ and $f = BD$, prove that

$$\frac{3R^2}{r^2} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} + \frac{e}{f} + \frac{f}{e}.$$

Marius Olteanu, Râmnicu Vâlcea

L291. Denote by a the number of finite decimal fractions that are written, in ordinary form, as fractions of type $\frac{1}{x}$ (unit fractions) where x is a natural number of n digits (in base 10). Prove that $4n + 1 \leq a \leq 6n + 1$.

Cosmin Manea şi Dragoş Petrică, Piteşti

L292. Let $(F_n)_{n\geq 0}$ be Fibonacci's sequence and $(L_n)_{n\geq 0}$ be Lucas' sequence, defined by $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $\forall n \in \mathbb{N}^*$, respectively $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$, $\forall n \in \mathbb{N}^*$. If p, $j \in \mathbb{N}^*$ are arbitrary, show that

$$a) \sum_{k=0}^{n} \frac{C_n^k}{F_{k+j}^p} \geq \frac{2^{(p+1)n}}{F_{2n+j}^p}; \quad b) \sum_{k=0}^{n} \frac{C_n^k}{L_{k+j}^p} \geq \frac{2^{(p+1)n}}{L_{2n+j}^p};$$

Nicuşor Zlota, Focşani

L293. Let $a_1, \ldots, a_n \in (0, 1]$. Show that the following inequality holds:

$$\sum_{k=1}^{n} \frac{(1-a_k)^2}{a_k} \leq \frac{(1 - \prod_{k=1}^{n} a_k)^2}{\prod_{k=1}^{n} a_k}.$$

Marian Tetiva, Bârlad

L294. Show that any positive numbers a, b, c satisfy the inequality

$$a^3 + b^3 + c^3 - \sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq$$
$$\geq \frac{1}{2}|(a-b)(b-c)(c-a)|.$$

Marian Tetiva, Bârlad

L295. Determine the minimum value of the positive real number k so that any positive real numbers a, b, c satisfy the inequality

$$\sum \frac{a^2 + b^2}{a^2 + ab + b^2} + k\frac{ab + bc + ca}{a^2 + b^2 + c^2} \leq k + 2.$$

Florin Stănescu, Găeşti