L251. Fie $n \in \mathbb{N}$, $n \geq 2$, şi numerele reale nenegative $x_1, x_2, \ldots, x_n$ cu proprietatea că $x_1^2 + x_2^2 + \ldots + x_n^2 = 3n^2$. Demonstraţi că $(x_1 + x_2 + \ldots + x_n)^3 \geq 9n(x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n)$.

Lucian Tuţescu şi Ionuţ Ivănescu, Craiova

L252. Fie $n \in \mathbb{N}$, $n \geq 5$, şi numerele reale $a_1 < a_2 < \ldots < a_n$. Se calculează toate sumele $a_i + a_j$, $i \neq j$, obţinând t rezultate distincte. Demonstraţi că $t \geq 2n - 3$ şi că $t = 2n - 3$ dacă şi numai dacă $a_1, a_2, \ldots, a_n$ este progresie aritmetică.

Titu Zvonaru, Comăneşti

L253. Fie a, b, c trei numere reale pozitive cu $a \leq c$ şi $x, y, z \in [a, c]$ astfel încât $x + y + z = a + b + c$ şi $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Arătaţi că numerele x, y şi z coincid într-o anumită ordine, cu a, b şi c.

Marian Tetiva, Bârlad

L254. Determinaţi numerele reale x, y, z din intervalul $[1, 3]$ astfel încât $x^2 + y^2 + z^2 = 14$ şi $x^3 + y^3 + z^3 = 36$.

Marian Tetiva, Bârlad

L255. Se consideră numerele reale $a < c < b$ şi şirul $(x_n)_{n\geq 1}$. Orice subşir convergent al şirului (x_n) are limita a sau limita b. Notăm $A_n = \{k \in \mathbb{N} | k \leq n$ şi $x_k \leq c\}$ şi $B_n = \{k \in \mathbb{N} | k \leq n$ şi $x_k > c\}$. Dacă există şi este finită şi nenulă limita $L = \lim\limits_{n\to\infty} \frac{\operatorname{card} A_n}{\operatorname{card} B_n}$, arătaţi că şirul $y_n = \frac{x_1 + x_n + \ldots + x_n}{n}$ este convergent şi aflaţi limita sa (funcţie de a, b şi L). Studiaţi şi cazurile $L = 0$ şi $L = +\infty$.

Cristinel Mortici, Târgovişte

Training Problems for Mathematical Contests

A. Junior Highschool Level

G246. Two children, A and B, play a game. This takes place on a rectangular square consisting of $a \times b$ small squares, where a and b are odd natural numbers, each of them proposed by one of the two children. The players mark, successively and once at a time, a cell in the table as it follows: A begins the game by marking a small square (m, n), where m represents the row and n – the column of the marked cell. Then, B marks one of the cells $(m \pm 1, n \pm 3)$ or $(m \pm 3, n \pm 1)$, situated inside the big table. Every time when a player comes to his turn, he chooses an already marked position (p, q) and he is allowed to mark one of the positions $(p \pm 1, q \pm 3)$ or $(p \pm 3, q \pm 1)$ whick is still blank on the table. The player who has no more cell to mark when its turn comes up loses the game. Prove that A has a strategy for winning.

Silviu Boga, Iaşi

G247. Let $A = \{1, 2, 3, \ldots, n\}$, $n \geq 6$, and X, Y be two disjoint subsets of the set A, $X \cup Y = A$, each of them consisting of at least three elements. Prove that four elements $x, y \in X$, $x \neq y$ and $a, b \in Y$, $a \neq b$ exist such that $x - y = a - b$.

Gheorghe Iurea, Iaşi

G248. If $a \in \mathbb{N}^*$ show that the number $5a\,(a^{\,2}+1)$ is not a perfect square.

Gheorghe Iurea, Iaşi

G249. Solve in natural numbers the equation $85^{\,m} - n^{\,4} = 4$.

Cristinel Mortici, Târgovişte

G250. Prove that $a^3+b^3 \geq 2\sqrt{ab}(a-2\,b)(b-2\,a)$, any would be the positive real numbers a and b.

Gabriel Popa, Iaşi

G251. If a, b, c are positive real numbers such that $ab+bc+ca=3$, show that $a^2(b+c)+b^2(c+a)+c^2(a+b) \geq 6$.

Monica Golea, elevă, Craiova

G252. Let $n \in \mathbb{N}$, $n \geq 2$ and consider the positive real numbers $x_1, x_2, \ldots, x_n$. If $S = x_1 + x_2 + \ldots + x_n$, prove that

$$\max\left(x_1 + \frac{1}{S-x_1}, x_2 + \frac{1}{S-x_2}, \ldots, x_n + \frac{1}{S-x_n}\right) \geq \frac{2}{\sqrt{n-1}}.$$

Ani Drăghici şi Mariana Mărculescu, Craiova

G253. Let M be an arbitrary point on the side AB of the square $ABCD$. The angle bisector of $\widehat{MDC}$ intersects the side BC at point N. Show that $BM+BN < AM+CN$.

Cecilia Deaconescu, Piteşti

G254. The diagonals of the trapezium $ABCD$ cut each other at point O. The line through O that is parallel to the base AB intersect the side BC at P. The point Q is situated in the opposite half-plane to the one determine by line AD and point B, and the lines QB and QC intersect AD at R, respectively at S. Prove that the lines PQ, BS and CR are concurrent.

Claudiu-Ştefan Popa, Iaşi

G255. Let AB and AC be the tangents from point A to a circle $\mathcal{C}$ (B and C being the points of contact) and let $\mathcal{R}$ be the plane region closed by the small arc $\widehat{BC}$ of circle $\mathcal{C}$ and the line segments AB and AC. Prove that $MN \leq \max(AB, AC)$, any would be the points M and N in $\mathcal{R}$.

Marian Tetiva, Bârlad

B. Highschool Level

L246. Let us consider the triangle ABC with $m(\widehat{A}) \geq 90°$, which is inscribed in the circle $\mathcal{C}$. The points D and D' are taken on the side BC such that $\widehat{ABC} \equiv \widehat{CAD}$ and $\widehat{ACB} \equiv \widehat{BAD'}$. The tangent circle to the lines AD, BD and to the circle $\mathcal{C}$ is also tangent to the line segment BD at M. The circle tangent to the lines AD', CD' and to circle $\mathcal{C}$ is tangent to the line segment CD' at N. Show that $\dfrac{MN}{BC} \leq \sqrt{2}-1$.

Neculai Roman, Mirceşti (Iaşi)

L247. Three points are consideered on the sides BC, CA and AB of the triangle ABC, respectively denoted A_1, B_1 and C_1, such that $AB + BA_1 = AC + CA_1$,

$AB + AB_1 = BC + CB_1$ and $AC + AC_1 = BC + BC_1$. If A_2, B_2 and C_2 are the contact points of the circle inscribed in triangle ABC with the sides BC, CA and respectively AB, show that $A_1 B_1^2 + B_1C_1^2 + C_1A_1^2 \geq A_2 B_2^2 + B_2 C_2^2 + C_2 A_2^2$.

Marius Olteanu, Rm. Vâlcea

L248. Prove that the following inequality holds in any triangle:

$$\frac{3(r_a + r_b + r_c)}{2p^2} \geq \frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} \geq \frac{6}{5R + 2r}.$$

Andi Gabriel Brojbeanu, elev, Târgovişte

L249. Let $ABCD$ be a bicentric quadrilateral, and let e and f denote the lengths of its diagonals. Prove that $\dfrac{p}{\sqrt{2\sqrt{2}Rr}} \geq \dfrac{e+f}{\sqrt{ef}}$.

Vasile Jiglău, Arad

L250. Establish which numbers n among $1, 2, \ldots, 9$ satisfy the equation $\operatorname{tg}\dfrac{\pi}{n} - \operatorname{tg}\dfrac{2\pi}{n} + \operatorname{tg}\dfrac{4\pi}{n} = \sqrt{3n}$.

Ionel Tudor, Călugăreni

L251. Let $n \in \mathbb{N}$, $n \geq 2$ and consider the nonnegative numbers $x_1, x_2, \ldots, x_n$ with the property that $x_1^2 + x_2^2 + \ldots + x_n^2 = 3n^2$. Prove that $(x_1 + x_2 + \ldots + x_n)^3 \geq 9\,n\,(x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n)$.

Lucian Tuţescu şi Ionuţ Ivănescu, Craiova

L252. Let $n \in \mathbb{N}$, $n \geq 5$ and the real numbers $a_1 < a_2 < \ldots < a_n$. All the sums $a_i + a_j$, $i \neq j$ are calculated, giving distinct results. Prove that $t \geq 2\,n - 3$ and that $t = 2\,n - 3$ if and only if $a_1, a_2, \ldots, a_n$ is an arithmetic progression.

Titu Zvonaru, Comăneşti

L253. Let a, b, c be positive real numbers with $a \leq c$ and x, y, $z \in [a, c]$ such that $x + y + z = a + b + c$ and $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$. Show that the numbers x, y and z respectively coincide, in a certain order, with a, b and c.

Marian Tetiva, Bârlad

L254. Find the real numbers x, y, z in $[1, 3]$ such that $x^2 + y^2 + z^2 = 14$, and $x^3 + y^3 + z^3 = 36$.

Marian Tetiva, Bârlad

L255. The real numbers $a < c < b$ and the sequence $(x_n)_{n\geq 1}$ are considered. Any convergent subsequence of sequence (x_n) has limit a or limit b. Let us denote $A_n = \{k \in \mathbb{N} \,|\, k \leq n \text{ and } x_k \leq c\}$ and $B_n = \{k \in \mathbb{N} \,|\, k \leq n \text{ and } x_k > c\}$. If there exists the limit $L = \lim\limits_{n\to\infty} \dfrac{\operatorname{card} A_n}{\operatorname{card} B_n}$ and it is finite and nonzero, show that the sequence $y_n = \dfrac{x_1 + x_n + \ldots + x_n}{n}$ converges and find its limit (as a function of a, b and L). Study the cases $L = 0$ and $L = +\infty$ as well.

Cristinel Mortici, Târgovişte