L209. Se consideră triunghiul ABC şi punctele M, N, P, Q, R, S definite prin $\overrightarrow{BM} = k \cdot \overrightarrow{MC}$, $\overrightarrow{CN} = k \cdot \overrightarrow{NA}$, $\overrightarrow{AP} = k \cdot \overrightarrow{PB}$, $\overrightarrow{AM} = p \cdot \overrightarrow{MQ}$, $\overrightarrow{BN} = p \cdot \overrightarrow{NR}$, $\overrightarrow{CP} = p \cdot \overrightarrow{PS}$, unde $k, p \in \mathbb{R}^* \setminus \{-1\}$. Demonstraţi că $S_{MNP} \geq \frac{1}{4} \cdot S_{ABC}$, iar $S_{QRS} \geq \left(\frac{p+3}{2p}\right)^2 \cdot S_{ABC}$.

Marius Olteanu, Rm. Vâlcea

L210. Cercul A-exînscris triunghiului ABC este tangent prelungirilor laturilor AB şi AC în P, respectiv Q. Bisectoarele exterioare ale unghiurilor B şi C intersectează dreapta PQ în S respectiv T. Demonstraţi că $PQ \leq ST + BC$.

Titu Zvonaru, Comăneşti

L211. Arătaţi că $\dfrac{\sin^3 x}{(1+\sin^2 x)^2} + \dfrac{\cos^3 x}{(1+\cos^2 x)^2} \leq \dfrac{3\sqrt{3}}{16}$, $\forall x \in \mathbb{R}$.

Mihàly Bencze, Braşov

L212. Demonstraţi că $\dfrac{3}{2} + \sum \dfrac{ab}{a^2+b^2} \leq \sum \dfrac{(ab+c^2)^2}{(a^2+c^2)(b^2+c^2)}$ (sumele fiind ciclice) pentru orice numere reale a, b, c printre care nu se găsesc două egale cu 0.

Marian Tetiva, Bârlad

L213. Fie $m_1, \ldots, m_k$ numere naturale nenule şi α un număr iraţional.

a) Arătaţi că există $x_1, \ldots, x_k \in \mathbb{N}^*$ astfel încât $\dfrac{[x_1\alpha]}{m_1} = \ldots = \dfrac{[x_k\alpha]}{m_k}$.

b) Arătaţi că există $y_1, \ldots, y_k \in \mathbb{N}^*$ astfel încât $m_1[y_1\alpha] = \ldots = m_k[y_k\alpha]$.

Marian Tetiva, Bârlad

L214. Fie $A \in \mathcal{M}_n(\mathbb{R})$ o matrice simetrică al cărei polinom caracteristic este X^n. Arătaţi că A este matricea nulă.

Marian Tetiva, Bârlad

L215. Avem la dispoziţie $2n+1$ pietricele ($n \geq 1$) astfel încât orice submulţime de $2n$ pietricele poate fi împărţită în două grămezi de câte n pietricele având aceeaşi masă totală. Demonstraţi că toate pietricelele au aceeaşi masă.

Adrian Reisner, Paris

Training problems for mathematical contests

A. Junior highschool level

G206. How many subsets of the set $M = \{1, 2, 3, \ldots, 100\}$ have 50 elements and do not contain any pair of successive numbers ?

Gheorghe Iurea, Iaşi

G207. Show that the number $N = 2^{2009} + 3^{2010} + 4^{2011}$ is not a perfect square.

Andrei Eckstein, Timişoara

G208. Show that the equation $x^2 + y^2 = z(x+y+1)$ has infinitely many solutions in the set of natural numbers.

Cosmin Manea and Dragoş Petrică, Piteşti

G209. Solve, in the set of natural numbers, the equation $4\,abc = (a+2)(b+1)(c+1)$.

Titu Zvonaru, Comăneşti

G210. Prove that that the fraction $\dfrac{a^{3n+2}-a^{3n+1}+(-1)^n}{a^{3n+8}-a^{3n+7}+(-1)^n}$ is reducible for any numbers $a,n\in\mathbb{N}$, $a\geq 2$.

Dan Popescu, Suceava

G211. Show that the expression

$$E = y_1\left(\frac{x_2(a_1+a_2)+x_3a_1}{x_1+x_2+x_3}\right)^2 + y_2\left(\frac{x_1(a_1+a_2)+x_3a_2}{x_1+x_2+x_3}\right)^2 + \\ + y_3\left(\frac{x_1a_1-x_2a_2}{x_1+x_2+x_3}\right)^2 - \frac{1}{y_1+y_2+y_3}\cdot \\ \cdot\left[-y_1\frac{x_2(a_1+a_2)+x_3a_1}{x_1+x_2+x_3}+y_2\frac{x_1(a_1+a_2)+x_3a_2}{x_1+x_2+x_3}+y_3\frac{x_1a_1-x_2a_2}{x_1+x_2+x_3}\right]^2,$$

where $a_i,x_i,y_i\in\mathbb{R}_+^*(i=1,2,3)$, does not depend of x_1,x_2,x_3.

Mircea Bîrsan, Iaşi

G212. Let ABC be a triangle with $m(\widehat{B})=135^\circ$ and $m(\widehat{C})=30^\circ$. Determine the measures of the angles of triangle ABD, where D is the symmetric of point C with respect to point B.

Eugeniu Blăjuţ, Bacău

G213. Let ABC be a triangle with the property that two points M and N exist in its interior such that $BN=CM$ and $\triangle ABM\sim\triangle ACN$. Show that $AB=AC$.

Crisitan Lazăr, Iaşi

G214. The isosceles triangle ABC is considered, with $AB=AC$ and $m(\widehat{A})<90^\circ$. We build the altitude line CF and let E be the midpoint of the segment BF, while D is a point on the segment BC. If $\widehat{ADE}\equiv\widehat{B}$, show that D is the midpoint of segment BC.

Claudiu Ştefan Popa and Gabriel Popa, Iaşi

G215. In the parallel planes P_1 and P_2 the circles $\mathcal{C}_1=\mathcal{C}(O_1,R_1)$, respectively $\mathcal{C}_2=\mathcal{C}(O_2,R_2)$ are considered. Let $\mathcal{K}_1$ be the cone of vertex O_2 and base $\mathcal{C}_1$, and $\mathcal{K}_2$ $-$the cone of vertex O_1 and base $\mathcal{C}_2$. Show that the intersection of the two cones is a circle and find the position of its center as well as the length of its radius.

Temistocle Bîrsan, Iaşi

B. Highschool Level

L206. Let P be a point on the median from A of the triangle ABC. The parallel through P to AC cuts AB at point M, and the symmetric point of P with respect to the mipoint of AC is N. Show that $MN\parallel BC$ if and only if P is centroid of triangle ABC.

Silviu Boga, Iaşi

L207. Let $ABCD$ be a convex quadrilateral and let M,N,P be points on the line segments AB,CD and respectively BC such that $\dfrac{MB}{AB}=\dfrac{ND}{DC}=\dfrac{BP}{BC}=k$. If R

and S are the midpoints of segments AP, respectively MN, calculate (as a function of k) the ratio $\dfrac{RS}{AD}$.

Titu Zvonaru, Comăneşti

L208. A right circular cone of axis d and radius R_1, and a sphere of center O and radius R_2, are exterior-tangent at the point A. Let B be the symmetric of A with respect to d and let π be the plane that passes through B, is perpendicular on the plane determined by O and d and forms an angle of $30°$ with axis d. Calculate the ratio between the radii of the two surfaces knowing that their sections through the plane π have equal areas.

Temistocle Bîrsan, Iaşi

L209. It is considered the triangle ABC and the points M, N, P, Q, R, S defined by $\overrightarrow{BM} = k\cdot\overrightarrow{MC}$, $\overrightarrow{CN} = k\cdot\overrightarrow{NA}$, $\overrightarrow{AP} = k\cdot\overrightarrow{PB}$, $\overrightarrow{AM} = p\cdot\overrightarrow{MQ}$, $\overrightarrow{BN} = p\cdot\overrightarrow{NR}$, $\overrightarrow{CP} = p\cdot\overrightarrow{PS}$, where $k, p \in \mathbb{R}^*\backslash\{-1\}$. Prove that $S_{MNP} \geq \dfrac{1}{4}\cdot S_{ABC}$, and $S_{QRS} \geq \left(\dfrac{p+3}{2p}\right)^2 \cdot S_{ABC}$.

Marius Olteanu, Rm. Vâlcea

L210. The A-excircle to the triangle ABC is tangent to the prolongations of the sides AB and AC at P, respectively Q. The exterior angle bisectors of B and C intersect the line PQ at S, respectively T. Prove that $PQ \leq ST + BC$.

Titu Zvonaru, Comăneşti

L211. Show that $\dfrac{\sin^3 x}{(1+\sin^2 x)^2} + \dfrac{\cos^3 x}{(1+\cos^2 x)^2} \leq \dfrac{3\sqrt{3}}{16}$, $\forall x \in \mathbb{R}$.

Bencze Mihàly, Braşov

L212. Show that $\dfrac{3}{2} + \sum \dfrac{ab}{a^2+b^2} \leq \sum \dfrac{(ab+c^2)^2}{(a^2+c^2)(b^2+c^2)}$ (the sums being cyclic) for any real numbers a, b, c with no two numbers equal to 0 among them.

Marian Tetiva, Bârlad

L213. Let $m_1, \ldots, m_k$ be nonzero natural numbers and α an irrational number.

a) Show that $x_1, \ldots, x_k \in \mathbb{N}^*$ exist such that $\dfrac{[x_1\alpha]}{m_1} = \ldots = \dfrac{[x_k\alpha]}{m_k}$.

b) Show that $y_1, \ldots, y_k \in \mathbb{N}^*$ exist such that $m_1[y_1\alpha] = \ldots = m_k[y_k\alpha]$.

Marian Tetiva, Bârlad

L214. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix whose characteristic polynomial is X^n. Show that A is the null matrix.

Marian Tetiva, Bârlad

L215. We have at our disposal $2n+1$ small stones ($n \geq 1$) such that any subset of small stones can be divided into two heaps of n small stones each and having the same total sum. Prove that all the small stones have the same mass.

Adrian Reisner, Paris