L192. Pe $\mathcal{P}$ mulţimea polinoamelor unitare având coeficienţii în intervalul $[0,9]$. Determinaţi supremumul modulelor rădăcinilor complexe ale polinoamelor din $\mathcal{P}$.

Adrian Reisner, Paris

L193. Fie $A \in M_3(\mathbb{R})$ o matrice simetrică, având urma egală cu determinantul. Demonstraţi că $2 \cdot Tr(A^{n+2}) \geq Tr\, A \cdot (Tr(A^{n+1}) - Tr(A^{n-1}))$, $\forall n \in \mathbb{N}^*$.

Paul Georgescu şi Gabriel Popa, Iaşi

L194. Fie $p \geq 2$ un număr natural şi ecuaţia $\dfrac{1}{\sqrt[p]{1+x}} + \dfrac{1}{\sqrt[p]{2+x}} + \ldots + \dfrac{1}{\sqrt[p]{n+x}} = \sqrt[p]{x^{p-1}}$. Demonstraţi că, pentru fiecare $n \in \mathbb{N}^*$, ecuaţia are o unică soluţie pozitivă x_n, arătaţi că şirul $(x_n)_{n\geq 1}$ este monoton şi calculaţi $\lim\limits_{n\to\infty} \dfrac{n}{x_n}$.

Gabriel Dospinescu, student, Paris

L195. Fie $(a_n)_{n\geq 1}$ un şir de numere reale mai mari ca 1, astfel încât $a_{n+1} \geq 2a_n - 1$, $\forall n \in \mathbb{N}^*$. Definim $x_n = \dfrac{1}{a_1} + \dfrac{1}{a_1 a_2} + \ldots + \dfrac{1}{a_1 a_2 \ldots a_n}$, $\forall n \geq 1$.

a) Demonsraţi că şirul $(x_n)_{n\geq 1}$ este convergent (notăm cu x limita sa).

b) Arătaţi că şirul $(\alpha_n)_{n\geq 1}$, unde $x = x_n + \dfrac{\alpha_n}{a_1 a_2 \ldots a_n (a_{n+1} - 1)}$, $\forall n \geq 1$, este monoton, convergent şi determinaţi limita sa.

Dumitru Mihalache şi Marian Tetiva, Bârlad

Training problems for mathematical contests

A. Junior highschool level

G186. Let m, n, x, y be four nonzero natural numbers such that $m < n$, $\dfrac{m+n}{(m,n)}$ is an even number, and $x^m = y^n$. Prove that $x - y$ is not a perfect square.

Geanina Hăvârneanu, Iaşi

G187. Let us consider the natural numbers a, b, c, d and x such that ac, bd and $ad + bc$ are divisible by x. Show that bc is divisible by x.

Ciprian Baghiu, Iaşi

G188. Determine the triples $(p, q, r) \in \mathbb{Z}^3$ such that

$$[a^{p\,q} + b^{p\,q} + (a^p + b^p)^{\,q}\,]^r = r\,[a^{p\,q\,r} + b^{p\,q\,r} + (a^p + b^p)^{q\,r}],\ \forall a, b \in \mathbb{R}.$$

Temistocle Bîrsan, Iaşi

G189. Three spheres are considered with their volumes v_1, v_2, v_3, their areas s_1, s_2, s_3 and the lengths of their equatorial circles l_1, l_2, l_3. Prove that

$$\frac{v_1}{l_1 + l_2} + \frac{v_2}{l_2 + l_3} + \frac{v_3}{l_3 + l_1} \geq \frac{1}{12\pi}(s_1 + s_2 + s_3).$$

D.M. Bătineţu-Giurgiu, Bucureşti

G190. If x and y are irrational numbers such that x^2+y, y^2+x and $x+y$ are rational, prove that $xy < \frac{1}{4}$.

Neculai Moraru, Suceava and Silviu Boga, Iaşi

G191. Let $A = \{1, 2, \ldots, n\}$ with $n \in \mathbb{N}, n \geq 5$. Prove that the set A can be written as the union of two disjoint sets B and C so that the product of the elements of B be equal to the sum of elements of C.

Gheorghe Iurea, Iaşi

G192. Prove that, in any non-isosceles triangle ABC, six points $M, N \in [BC]$, $P, Q \in [AB]$ and $R, S \in [AC]$ exist such that the segments $[AU], [BT]$ and $[CV]$ are the sides of an acute triangle, any would be the points $U \in [MN], V \in [PQ]$ and $T \in [RS]$.

Marius Drăgan, Bucureşti

G193. Effectively using 24 matches, build a polygon with its area equal to 7 units, where a unit area is the area of a square with one match stick side. (*Related to a problem from the book "Mathematical Amusements" by Martin Gardner*).

Dumitru Mihalache, Bârlad

G194. Let $ABCD$ be a convex quadrilateral, and let the half lines $[CK$ and $[CL$ be the angle-bisectors of the angles $\widehat{ACD}$ and respectively $\widehat{ACB}$ $(K \in AD, L \in AB)$. Prove that the line LK passes through the centre of the circle inscribed in the triangle ABD if and only if the quadrilateral $ABCD$ can be inscribed into a circle.

Neculai Roman, Mirceşti (Iaşi)

G195. The points M and P lie on the non-parallel side (AD) of the trapezoid $ABCD$, such that $M \in [AP]$. The parallel lines to the two bases intersect BC at N and at Q, respectively. Prove that the trapezia $ABNM$ and $PQCD$ have their diagonals respectively parallel if and only if CD and PQ are directly proportional to MN and AB (*Connected with problem G105 of RecMat 1/2006*).

Claudiu Ştefan Popa, Iaşi

B. Highschool Level

L186. Determine the measures of the angles of triangle ABC knowing that

$$\operatorname{tg} A + 2\sqrt{\operatorname{tg} B} + 3\sqrt[3]{\operatorname{tg} C} = 6\sqrt[6]{\operatorname{tg} A + \operatorname{tg} B + \operatorname{tg} C}.$$

Cătălin Calistru, Iaşi

L187. Let ABC be a triangle, and let the points $D, L \in (BC)$, $E, M \in (CA)$ and $F, N \in (AB)$. Denote $\{P\} = FE \cap AL$, $\{Q\} = DF \cap BM$, $\{S\} = ED \cap CN$. Prove that if two of the assertions below hold true the third one is also true:

(i) the straight lines AD, BE and CF are concurrent;
(ii) the straight lines AL, BM and CN are concurrent;
(iii) the straight lines DP, EQ and FS are concurrent.

Titu Zvonaru, Comăneşti

L188. Let Oxy be a Cartesian system of coordinates and consider the line d : $x = 1$ with the point $M \in d$. The point P is obtained from M by a rotation of centre O and angle 2α, where a is the directed angle $\widehat{yOM}$. Write the equation $y = f(x)$ of the curve described by P when M runs along the line d and draw the graph of this curve.

Temistocle Bîrsan, Iaşi

L189. A system of $n \geq 3$ coplanar points $A_1, A_2, \ldots, A_n$, with any three among them being not collinear, is said to be *good* when for any $1 \leq i < j < k \leq n$ and the points $M \in [A_iA_j]$, $N \in [A_jA_k]$ and $P \in [A_kA_i]$, the inequality $MN^2+NP^2+MP^2 \leq A_iA_j^2 + A_jA_k^2 + A_kA_i^2$ holds. Determine the good systems with a maximum number of points.

Vlad Emanuel, student, Bucureşti

L190. Prove that

$$8\sqrt{15} \leq (\sqrt{x} + \sqrt{x+48})(\sqrt{16-x} + \sqrt{1-x}) \leq 36, \quad \forall x \in [0,1].$$

Gheorghe Iurea, Iaşi

L191. Show that any three positive real numbers x, y, z satisfy the inequality

$$(x^2+y^2+z^2)\left(\frac{1}{(x+y)(x+z)} + \frac{1}{(x+y)(y+z)} + \frac{1}{(x+z)(y+z)}\right) \geq \frac{9}{4} \geq$$
$$\geq (xy+xz+yz)\left(\frac{1}{(x+y)(x+z)} + \frac{1}{(x+y)(y+z)} + \frac{1}{(x+z)(y+z)}\right).$$

Marian Tetiva, Bârlad

L192. Let $\mathcal{P}$ be the set of monic polynomials with their coefficients in the interval $[0,9]$. Determine the supremum of the modules of the complex roots of polynomials in $\mathcal{P}$.

Adrian Reisner, Paris

L193. Let $A \in M_3(\mathbb{R})$ be a symmetric matrix with its trace equal to its determinant. Prove that $2 \cdot Tr\,(A^{n+2}) \geq Tr\,A \cdot (Tr(A^{n+1}) - Tr(A^{n-1}))$, $\forall n \in \mathbb{N}^*$.

Paul Georgescu and Gabriel Popa, Iaşi

L194. Let $p \geq 2$ be a natural number and consider the equation $\dfrac{1}{\sqrt[p]{1+x}} + \dfrac{1}{\sqrt[p]{2+x}} + \ldots + \dfrac{1}{\sqrt[p]{n+x}} = \sqrt[p]{x^{p-1}}$. Prove that, for any $n \in \mathbb{N}^*$, this equation has a unique positive solution x_n, show that the sequence $(x_n)_{n\geq1}$ is monotonic and calculate the limit $\lim\limits_{n\to\infty} \dfrac{n}{x_n}$.

Gabriel Dospinescu, student, Paris

L195. Let $(a_n)_{n\geq1}$ be a sequence of real numbers that are greater than 1, such that $a_{n+1} \geq 2a_n - 1$, $\forall n \in \mathbb{N}^*$. We define $x_n = \dfrac{1}{a_1} + \dfrac{1}{a_1a_2} + \ldots + \dfrac{1}{a_1a_2\ldots a_n}$, $\forall n \geq 1$.

a) Prove that the sequence $(x_n)_{n\geq1}$ converges (and denote by x its limit).

b) Show that the sequence $(a_n)_{n\geq1}$, where $x = x_n + \dfrac{\alpha_n}{a_1a_2\ldots a_n(a_{n+1}-1)}$, $\forall n \geq 1$, is monotonic and convergent, and determine its limit.

Dumitru Mihalache and Marian Tetiva, Bârlad