b) Există funcţii $f : (a, b) \cap \mathbb{Q} \to \mathbb{R}$ cu proprietatea că $|f(x) - f(y)| \geq c$, $\forall x, y \in (a, b) \cap \mathbb{Q}$, unde c este o constantă pozitivă?

Geanina Hăvârneanu, Iaşi

L174. Fie $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ numere reale pozitive şi $a = \left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2$, $b = \left(\sum_{i=1}^{n} \sqrt{b_i}\right)^2$. Arătaţi că există $x_0 > 0$ astfel încât $\left[\sum_{i=1}^{n} \sqrt{a_i x + b_i}\right] - [\sqrt{ax + b}] \in \{0, 1\}$, $\forall x > x_0$.

Marian Tetiva, Bârlad

L175. Arătaţi că

$$\sum_{k=0}^{[\frac{n}{2}]} C_n^{2k} \Omega_k = 2^n \Omega_n, \quad n \in \mathbb{N},$$

unde $\Omega_k = \dfrac{(2k-1)!!}{(2k)!!}$, $k \in \mathbb{N}^*$ (se convine ca $\Omega_0 = 1$).

Gheorghe Costovici, Iaşi

Training problems for mathematical contests

A. Junior highschool level

G166. Prove that the following assertions are true:

a) $\forall n \in \mathbb{N}$, $n \geq 2$, $\exists\, x_1, x_2, \ldots, x_n \in \mathbb{N}^*$ such that $x_1x_2 + x_2x_3 + x_nx_1 = x_1x_2 \ldots x_n$.

b) $\forall n \in \mathbb{N}$, $n \geq 5$, $\nexists\, x_1, x_2, \ldots, x_n \in 2\,\mathbb{N}^*$ such that $x_1x_2 + x_2x_3 + x_nx_1 = x_1x_2 \ldots x_n$.

c) $\exists\, x_1, x_2, \ldots, x_n \in \mathbb{N}^*$ such that $x_1x_2 + x_2x_3 + x_nx_1 = x_1x_2 \ldots x_n \Leftrightarrow n \in 2\,\mathbb{N}^* + 1$.

Dan Popescu, Suceava

G167. Let $d_1 < d_2 < \ldots d_k = n$ be all the positive divisors of the natural number n. Assuming that the subscripts i, j with $j > i > 13$ exist such that $d_7^2 + d_i^2 = d_j^2$, show that n is a multiple of 8.

Titu Zvonaru, Comăneşti

G168. For any x, $y, z \in \mathbb{R}_+^*$, prove that the following inequality holds:

$$\frac{x(y+z)}{x+y\,z} + \frac{y(x+z)}{y+x\,z} + \frac{z(x+y)}{z+x\,y} \leq 2\left(\frac{x^2}{x+y\,z} + \frac{y^2}{y+x\,z} + \frac{z^2}{z+x\,y}\right).$$

Ştefan Gavril, Piatra Neamţ

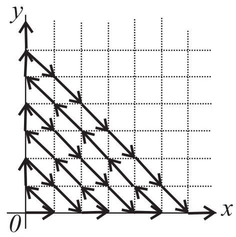
G169. Prove that an infinity of irrational numbers α exist with the property that α^3 and $\alpha^2 + \alpha$ are irrational numbers as well.

Gabi Ghidoveanu and Dumitru Mihalache, Bârlad

G170. A subset $A \subset \mathbb{R}$, of cardinal number 2009, has the property that each of its elements is greater than one tenth of the sum of the remaining 2008 numbers. Show that A contains at least 12 negative numbers.

Andrei Nedelcu, Iaşi

G171. The points in the plane whose both coordinates, in a Cartesian system of coordinates, are natural numbers, visited along the path indicated by the arrows in the figure at right, starting from the origin. We denote by $a_{n,k}$ the position of the point of coordinates (n,k) in the sequence thus obtained (for instance, $a_{0,0}=1$, $a_{0,2}=4$, $a_{2,2}=13$, etc.). Express the number $a_{n,k}$ in terms of n and k.



Lucian Georges Lăduncă, Iaşi

G172. A rectangular board of size $m \times n$, $m, n \geq 2$, has unit squares at the intersections of odd-order rows with odd-order columns, colored in black while the other squares remain white. To re-color a row (column) means to change the color of all the squares on that row (column). Show that the board cannot be turned to a completely white board by recoloring a couple of rows and columns.

Răzvan Ceucă, school student, Iaşi

G173. We denote by $T(a,b,c)$ the triangle whose side lengths are a, b, c. Prove that the triangles $T(b, 2c, 2m_b)$ and $T(c, 2b, 2m_c)$ can be successively manufactured from a single (the same) cardboard sheet without losses of material.

Petru Asaftei, Iaşi

G174. The isosceles triangle ABC is considered with $m(\widehat{A}) = 40^0$. Show that no points $P \in$Int(ABC) exist such that $m(\widehat{PAB}) = 30^0$, $m(\widehat{PBC}) = 10^0$ and $m(\widehat{PCA}) = 35^0$.

Gabriel Popa and Paul Georgescu, Iaşi

G175. Let $ABCD$ be a quadrilateral inscribed in the circle of radius R. Prove that $AB \cdot AD + CB \cdot CD \leq 4R^2$.

Gheorghe Costovici, Iaşi

B. Highschool level

L166. Let $ABCD$ be a rectangle, and C a circle passing through A which intersects the lines (AB), (AC) and (AD) at the points M, N and P, respectively. Show that $AM \cdot AB + AP \cdot AD = AN \cdot AC$.

Gheorghe Iurea, Iaşi

L167. Let ABC be a triangle with $AB > AC$. The circle inscribed in the triangle is tangent to the sides BC and AC at D and respectively E. We consider a point T on the side $[BC]$ and denote by J the center of the circle inscribed in $\triangle ABT$. If DE passes through the midpoint of the segment $[CJ]$, prove that the triangle ATC is isosceles.

Titu Zvonaru, Comăneşti

L168. Prove that in any triangle, with the usual notations, the following inequality holds:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{11\,p^2-15\,r^2-60\,R\,r}{6\,p^2-6\,r^2-24\,R\,r}\geq\frac{3}{2}\,.$$

Marius Olteanu, Rm. Vâlcea

L169. What is the probability that the radii of the excircles to a randomly chosen triangle be the sides of a new triangle?

Petru Minuţ, Iaşi

L170. Let $n\in\mathbb{N}$, $n\geq 2$ and $a_1,a_2,\ldots,a_n\in\mathbb{R}_+$ with $a_1+a_2+\cdots+a_n=S$. We consider $k\in\mathbb{N}$, $1\leq k\leq n-1$ and $\alpha_1,\alpha_2\in\mathbb{R}$ with $\alpha_1+\alpha_2=1$. Prove the inequality

$$\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}\left(a_{i_1}+a_{i_2}+\cdots+a_{i_k}\right)^{\alpha_1}\left(S-a_{i_1}-\cdots-a_{i_k}\right)^{\alpha_2}\leq$$
$$\leq\frac{k^{\alpha_1}\,(n-k)^{\alpha_2}}{n}\binom{n}{k}S.$$

(*Connected with 6117 of R.M.T. 1/1987*).

Paul Georgescu and Gabriel Popa, Iaşi

L171. For any x, $y\in\mathbb{R}_+^*$, prove that the following inequality holds:

$$\sqrt{x}+3\sqrt{2(x+y)}+\sqrt{y}\leq 2\left(\sqrt{3\,x+y}+\sqrt{x+3\,y}\right).$$

Marian Tetiva, Bârlad

L172. Let $P\in\mathbb{Q}[X]$ with $\deg P=n\geq 1$. If P admits a complex root a with its multiplicity m such that $n<2\,m$, prove that $a\in\mathbb{Q}$.

Adrian Reisner, Paris

L173. *a*) Do exist functions $f:(a,b)\to\mathbb{R}$ with the property that $|f(x)-f(y)|\geq c$ for $\forall x,y\in(a,b)$, where c is a positive constant?

b) Do exist functions $f:(a,b)\cap\mathbb{Q}\to\mathbb{R}$ with the property that $|f(x)-f(y)|\geq c$ for $\forall x,y\in(a,b)\cap\mathbb{Q}$, where c is a positive constant?

Geanina Hăvârneanu, Iaşi

L174. Let $a_1,a_2,\ldots,a_n$, $b_1,b_2,\ldots,b_n$ be positive real numbers and let $a=\left(\sum_{i=n}^{n}\sqrt{a_i}\right)^2$, $b=\left(\sum_{i=n}^{n}\sqrt{b_i}\right)^2$. Show that $x_0>0$ exists such that

$$\left[\sum_{i=n}^{n}\sqrt{a_ix+b_i}\right]-\left[\sqrt{ax+b}\right]\in\{0,1\}\,,\ \text{for}\ \forall x>x_0.$$

Marian Tetiva, Bârlad

L175. Show that

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}C_n^{2k}\Omega_k=2^n\,\Omega_n$$

where $\Omega_k=\dfrac{(2k-1)!!}{(2k)!!}$, $k\in\mathbb{N}^*$; by convention $\Omega_0=1$.

Gheorghe Costovici, Iaşi