L153. Găsiţi toate funcţiile $f:\mathbb{R}\to\mathbb{R}$ cu proprietatea că
$$f\left(x^2+xy+yf(y)\right)=xf(x+y)+f^2(y),\quad \forall x,y\in\mathbb{R}.$$
Adrian Zahariuc, student, Princeton

L154. Fie $P\in\mathbb{R}[X]$ un polinom de gradul n şi $p:\mathbb{R}\to\mathbb{R}$ funcţia polinomială asociată. Ştiind că mulţimea $\{x\in\mathbb{R}\mid p(x)=0\}$ are k elemente (distincte), iar funcţia $f:\mathbb{R}\to\mathbb{R}$, $f(x)=|p(x)|$ este derivabilă pe $\mathbb{R}$, arătaţi că numărul maxim de rădăcini complexe nenule ale lui P este egal cu $2\left[\frac{n}{2}\right]-2k$.

Vlad Emanuel, student, Bucureşti

L155. Fie $A,B\in\mathcal{M}_2(\mathbb{C})$ două matrice astfel încât matricea $AB-BA$ să fie inversabilă. Să se arate că urma matricei $(I_2+AB)(AB-BA)^{-1}$ este egală cu 1.

Florina Cârlan şi Marian Tetiva, Bârlad

Training problems for mathematical contests

A. Junior highschool level

G146. Let $x,y,z\in(0,\infty)$ such that $xyz=1$. Prove that
$$\frac{xy^3}{x^4+y+z}+\frac{yz^3}{y^4+z+x}+\frac{zx^3}{z^4+x+y}\geq 1.$$
Liviu Smarandache and Lucian Tuţescu, Craiova

G147. Let $n\in\mathbb{N}$, $n\geq 2$ be a fixed number and let a, b, c be natural numbers such that $na+(n+1)b+2nc=n^2+1$. Show that $n-\left[\frac{n-1}{2}\right]\leq a+b+c\leq n$.

Gheorghe Iurea, Iaşi

G148. Let $\overline{a_1a_2\ldots a_p}\in\mathbb{N}$. Show that every natural number has a multiple of the form $\overline{a_1a_2\ldots a_pa_1a_2\ldots a_p\ldots a_1a_2\ldots a_p0\ldots 0}$.

Marian Panţiruc, Iaşi

G149. *a*) Determine two prime numbers p, q so that $p<q$, and p^2-1 has more natural divisors than q^2-1.

b) Determine all the prime numbers p such that p^2-1 has exactly eight natural divisors.

Dan Popescu, Suceava

G150. Let m and n be nonzero natural numbers with the property that $m\leq 1+2+\cdots+n$. Show that m may be written as the sum of a couple of distinct numbers among 1, 2,..., n.

Marian Tetiva, Bârlad

G151. The bases of a prism are polygons with 2008 vertices. We number by $1,2,\ldots,2008$ the vertices of the lower basis and by a_1, a_2,..., a_{2008} the vertices of the upper basis, where $\{a_1,a_2,\ldots,a_{2008}\}=\{1,2,\ldots,2008\}$.

a) Show that we can find a numbering for the upper basis so that $i+a_i\,\vdots\,8$, $\forall i\in\{1,2,\ldots,2008\}$.

b) Show that we cannot find a numbering for the upper basis so that $i + a_i \vdots 9$, $\forall i \in \{1, 2, \ldots, 2008\}$.

Gabriel Popa and Gheorghe Iurea, Iaşi

G152. In the isosceles triangle ABC ($AB = AC$), B', C' denote the feet of the altitudes from B, respectively C. If $AB = 2\,B'C'$, determine the angles of the triangle.

Nela Ciceu, Bacău and Titu Zvonaru, Comăneşti

G153. In the triangle ABC, M is the midpoint of the side $[BC]$, $m(\widehat{ABC}) = 30°$ and $m(\widehat{ACB}) = 105°$. The perpendicular from C on AM cuts AB at Q. Calculatre the value of the ratio $\dfrac{QA}{QB}$.

Neculai Roman, Mirceşti (Iaşi)

G154. Let D be the midpoint of the side $[BC]$ in the equilateral triangle ABC of side length 1, and let P be a moving point on $[CD]$. Denote by M and N the projections of the points B, respectively C on AP Find the area of the geometric locus described by the segment $[MN]$.

Mariu Olteanu, Rm. Vâlcea

G155. Let $\mathcal{C}$ be the circumcircle of the acute-angled triangle $\triangle ABC$. Denote by P the intersection point of the tangents to the circle at B and C, $\{D\} = AP \cap \mathcal{C}$, while M and N are the midpoints of the small arc $\overset{\frown}{BC}$, respectively of the big arc $\overset{\frown}{BC}$. Show that the straight lines AM, DN and BC meet at a point.

Gabriel Popa, Iaşi

B. Highschool level

L146. The straight lines d_1, d_2,..., d_{n+1}, are considered in the plane such that any two lines are not parallel. We denote by $\alpha_k = m(\widehat{d_k, d_{k+1}})$, $\alpha_k \leq 90°$, $k = \overline{1, n}$. A segment of length 2 is cosidered on d_1 that is projected on d_2, then the obtained segment is projected on d_3 and so on, until a segment of length 1 is obtained on d_{n+1}. Knowing that $\tan\left(\min\left\{\alpha_i \mid i = \overline{1, n}\right\}\right) = \sqrt{\sqrt[n]{4} - 1}$, determine the angles α_k, $k = \overline{1, n}$.

Cristian Săvescu, student, Bucureşti

L147. A convex polygon with n sides, $n \geq 4$, is considered such that any pair of diagonals are not parallel and and any three diagonals do not meet at other points except the vertices of the polygon. Let us denote by n_i the number of intersection points of the diagonals inside the polygon and by n_e the number of intersection points of the diagonals outside the polygon.

a) Show that exactly eight polygons exist such that the inequality $n_i > n_e$ is satisfied.

b) Show that exactly three polygons exist such that $n_i + n_e kn^2$, $k \in \mathbb{N}^*$.

Mihai Haivas, Iaşi

L148. A point D is considered on the side (AB) of the triangle ABC such that $AB = 4\,AD$. In the same halfplane as point C with respect to the side AB, we take a point P such that $\widehat{PDA} \equiv \widehat{ACB}$ and $PB = 2\,PD$. Prove that the quadrilateral $ABCP$ is inscriptible, that si it admits a circumscribed circle.

Nela Ciceu, Bacău and Titu Zvonaru, Comăneşti

L149. Determine the position of the point P on the directrix line of the parabola $\mathcal{P}$, so that the area of the triangle PT_1T_2 be minimum, where T_1 and T_2 are the contact points with $\mathcal{P}$ of the tangents drawn from P to $\mathcal{P}$.

Adrian Corduneanu, Iaşi

L150. Let us consider the tetrahedron $A_1A_2A_3A_4$, and a point P inside it. Denote by $A_{i\,j} \in (A_i\,A_j)$ the orthogonal projections of P on the edge(s) A_iA_j of the tetrahedron. Prove that

$$\mathcal{V}_{PA_{12}A_{13}A_{23}} + \mathcal{V}_{PA_{12}A_{14}A_{24}} + \mathcal{V}_{PA_{13}A_{14}A_{34}} + \mathcal{V}_{PA_{23}A_{24}A_{34}} \le \frac{1}{4}\mathcal{V}_{A_1A_2A_3A_4}.$$

When the equality is attained?

Marius Olteanu, Rm. Vâlcea

L151. Prove than no natural numbers n and k exist such that $\left[\left(2+\sqrt{3}\right)^{2n+1}\right] = \left[\left(4+\sqrt{15}\right)^{k}\right]$.

Cosmin Manea and Dragoş Petrică, Piteşti

L152. For $a, b, c \in \mathbb{R}$ and $x \in \mathbb{R}_+$, prove the inequality

$$\frac{9}{a^2+b^2+c^2} \le \frac{3\,(x+1)^2\,(a+b+c)^4}{\left[3(x^2+1)(a^2+b^2+c^2)+2x(a+b+c)^2\right](ab+bc+ca)^2} \le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

I. V. Maftei and Dorel Băiţan, Bucureşti

L153. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f\left(x^2+xy+yf\,(y)\right) = xf\,(x+y) + f^2\,(y)\,, \quad \forall x, y \in \mathbb{R}.$$

Adrian Zahariuc, student, Princeton

L154. Let $P \in \mathbb{R}[X]$ a polynomial of degree n and $p : \mathbb{R} \to \mathbb{R}$ its associated polynomial function. Knowing that the set $\{x \in \mathbb{R} \mid p\,(x) = 0\}$ consists of k (distinct) elements, and the function $f : \mathbb{R} \to \mathbb{R}, f\,(x) = |\,p\,(x)|$ is differentiable on $\mathbb{R}$, show that the maximum number of nonzero complex roots of P equals $2\left[\frac{n}{2}\right] - 2k$.

Vlad Emanuel, student, Bucureşti

L155. Let $A, B \in \mathcal{M}_2\,(\mathbb{C})$ be two matrices such that the matrix $AB - BA$ is invertible. Show that the trace of the matrix $(I_2 + AB)\,(AB - BA)^{-1}$ is equal to 1.

Florentina Cârlan and Marian Tetiva, Bârlad