L71. Fie $n \in \mathbb{N}$, $n \geq 2$ fixat. Să se determine cea mai tare inegalitate de forma

$$\sum_{k=1}^{n} \sqrt{a_k^2 + n^2 - 1} \leq m \cdot \sum_{k=1}^{n} a_k + M,$$

unde m, M nu depind de $a_1, a_2, \ldots, a_n$, valabilă pentru orice numere $a_1, a_2, \ldots, a_n$ pozitive şi cu produsul 1.

Gabriel Dospinescu, Bucureşti

L72. Fie a, b numere raţionale, pozitive, distincte, astfel încât $a^n - b^n \in \mathbb{Z}$ pentru o infinitate de numere naturale n. Să se arate că a şi b sunt întregi.

Gabriel Dospinescu, Bucureşti

L73. Fie $k \in \mathbb{N}^*$, $k \geq 3$. Să se determine $n \in \mathbb{N} \setminus \{0, 1\}$ pentru care

$$\sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_k}}} \geq \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_k}, \quad \forall a_1, a_2, \ldots, a_k \in [0, \infty).$$

Gabriel Popa şi Paul Georgescu, Iaşi

L74. Fie $n \in \mathbb{N}^*$, $n \geq 2$ şi $a, b \in \mathbb{R}$, $a < b$. Dacă $f : [a, b] \to \mathbb{R}$ este continuă şi $\int_a^b x^k f(x)\, dx = 0$ pentru $0 \leq k \leq n$, atunci f are cel puţin $n + 1$ zerouri distincte în (a, b).

Andrei Nedelcu, Iaşi

L75. Să se determine $n \in \mathbb{N}$ pentru care este adevărată inegalitatea

$$\cos \varphi < \frac{1}{\sqrt[8]{1 + n \sin^4 \varphi}}, \quad \forall \varphi \in \left(0, \frac{\pi}{2}\right].$$

Cătălin Calistru, Iaşi

Training problems for mathematical contests

Junior high school level

G66. Considering the set $A = \{1, n+1, 2n+1, \ldots, mn+1\}$, $m, n \in \mathbb{N}^*$, $m > n$, find the number of distinct values taken by the sum $a_1 + a_2 + \ldots + a_n$, when $a_1, a_2, \ldots, a_n \in A$.

Petru Asaftei, Iaşi

G67. Let $b \in \mathbb{N}$, $b \geq 2$. It is said that $G \in \mathbb{N}$ is *decomposable* if we can write G as a sum of two numbers such that their expansions in the basis b have the same sum of digits. Prove that there exist infinitely many numbers which are not decomposable.

Adrian Zahariuc, high school student, Bacău

G68. Let $N \in \mathbb{N}^*$. Prove that there is $n \in \mathbb{N}$ such that no factorial ends in n, $n + 1, \ldots, n + N$ zeros.

Iuliana Georgescu, Iaşi

G69. Let $E(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{Q}$, $x \in \mathbb{R}$. If $a + b + c$ is an integer, prove that there exist infinitely many integers n such that $E(n)$ is also an integer.

Gheorghe Iurea, Iaşi

G70. Prove that the equation $x^2 + y^2 + 3x + y - 707 = 0$ has no solutions in $\mathbb{Q}^2$.

Dan Popescu, Iaşi

G71. Let $(m,n) \in \mathbb{N}^2 \setminus \{(0,0)\}$. Prove that

$$\frac{a}{(m+n)\,a^2+mb^2+nc^2}+\frac{b}{(m+n)\,b^2+mc^2+na^2}+\frac{c}{(m+n)\,c^2+ma^2+nb^2}\leq$$
$$\leq\frac{1}{2\,(m+n)}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\quad \forall a,b,c\in(0,\infty)\,.$$

Titu Zvonaru, Comăneşti

G72. Let I be the incenter of a triangle ABC. The circle with diameter $[AI]$ meets the bisectors of $\widehat{B}$ and $\widehat{C}$ in M, respectively N. Prove that M and N lie on the line joining the midpoints of $[AB]$ and $[AC]$.

Doru Buzac, Iaşi

G73. Let $ABCD$ be a rectangle with center O. Let $N \in (AO)$, let M be the midpoint of $[AD]$ and let $\{P\} = MN \cap CD$, $\{E\} = OP \cap BC$. Prove that $NE \perp BC$.

Andrei Nedelcu, Iaşi

G74. Let us consider n points such that any given four are the vertices of a tethraedron with volume at most 1. Prove that there is a tethraedron with volume at most 27 which contains all n points in its interior.

Tudor Chirilă, high school student, Iaşi

G75. Let $A_1A_2\ldots A_n$ be a regular n-gon with side 1, $n \geq 4$. We consider P_1 on the side $[A_1A_2]$ such that $P_1A_1 = a \in (0,1)$. A ray of light is emitted from P_1 towards and is reflected by the sides $[A_2A_3]$, $[A_3A_4]$, ... , generating the incidence points P_2, ... , P_n (supposing that the ray never meets the vertices A_1, A_2, ... , A_n) such that $m(\widehat{A_2P_1P_2}) = \alpha \in \left(\frac{\pi}{n},\frac{2\pi}{n}\right)$. Find the minimal value of l such that P_l and P_{l+1} do not belong to adjacent sides.

Irina Mustaţă, high school student, Iaşi

High school level

L66. Let ABC be a given triangle and let D, D_a be the points in which the incircle, respectively the A-escribed circle are tangent to the side BC. Let also E_b, F_c be the points in which the B-escribed and C-escribed circles are tangent to the side AC, respectively to the side AB. Prove that D, D_a, E_b, F_c are concyclic if and only if $AB = AC$ or $m(\widehat{A}) = 90°$.

Temistocle Bîrsan, Iaşi

L67. The parallel lines t_1 and t_2 are tangent to the circle $\mathcal{C}$ with center O, the circle $\mathcal{C}_1$ with center O_1 is tangent to t_1 and $\mathcal{C}$ and the circle $\mathcal{C}_2$ with center O_2 is tangent to t_2, $\mathcal{C}$ and $\mathcal{C}_1$; $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ being exterior to each other. Prove that the angle $\widehat{O_1OO_2}$ is acute and find the minimum value of its measure.

Neculai Roman, Mirceşti (Iaşi)

L68. *a*) Prove that, for $x,y,z\in(0,\infty)$,

$$\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}\geq 1+\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}.$$

b) Using *a*), prove that

$$\sqrt{1+4\cdot\frac{R}{r}}\geq 1+\sqrt{\frac{p-a}{p-b}}+\sqrt{\frac{p-b}{p-a}}$$

for any given triangle with the usual notations.

Marian Tetiva, Bârlad

L69. Find $n\in\mathbb{N}$, $n\geq 3$, such that there are n blue points and n red points in the same plane, no three points being collinear, such that the interior of any triangle with blue vertices contains at least one red point and the interior of any triangle with red vertices contains at least one blue point.

Adrian Zahariuc, high school student, Bacău

L70. Let $k,p\in\mathbb{N}^*$ and let a $82k\times 2p$ rectangle which is completely covered with 7×5 and 6×4 rectangles with no superpositions. Prove that the number of squares (x,y) with side 1, x even, y odd, which are vertices of 7×5 rectangles equals the total number of 7×5 rectangles. (By a 7×5 rectangle we mean a rectangle with length 7 and height 5).

Marius Pachiţariu, high school student, Iaşi

L71. Let $n\in\mathbb{N}$, $n\geq 2$. Find the best constants m, M such that

$$\sum_{k=1}^{n}\sqrt{a_k^2+n^2-1}\leq m\cdot\sum_{k=1}^{n}a_k+M$$

for any $a_1,a_2,\ldots,a_n>0$ satisfying $a_1a_2\cdot\ldots\cdot a_n=1$.

Gabriel Dospinescu, Bucureşti

L72. Let $a,b\in\mathbb{Q}$, $a,b>0$, $a\neq b$ such that $a^n-b^n\in\mathbb{Z}$ for infinitely many $n\in\mathbb{N}$. Prove that $a,b\in\mathbb{Z}$.

Gabriel Dospinescu, Bucureşti

L73. Let $k\in\mathbb{N}^*$, $k\geq 3$. Find $n\in\mathbb{N}\setminus\{0,1\}$ such that

$$\sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_k}}}\geq\sqrt[n]{a_1a_2\cdot\ldots\cdot a_k},\quad\forall a_1,a_2,\ldots,a_k\in[0,\infty).$$

Gabriel Popa and Paul Georgescu, Iaşi

L74. Let $n\in\mathbb{N}^*$, $n\geq 2$ and $a,b\in\mathbb{R}$, $a<b$. If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $\int_a^b x^k f(x)\,dx=0$ for any $k\in\mathbb{N}$, $0\leq k\leq n$, then f has at least $n+1$ distinct zeros in (a,b).

Andrei Nedelcu, Iaşi

L75. Find $n\in\mathbb{N}$ such that

$$\cos\varphi<\frac{1}{\sqrt[8]{1+n\sin^4\varphi}},\quad\forall\varphi\in\left(0,\frac{\pi}{2}\right].$$

Cătălin Calistru, Iaşi