L321. Notăm cu $[x]$ şi $\{x\}$ partea întreagă respectiv partea fracţionară ale numărului real x. Demonstraţi că, pentru orice număr real pozitiv x, are loc inegalitatea

$$\frac{[x]^2}{7[x]^2+(2\{x\}+[x])^2}+\frac{\{x\}^2}{7\{x\}^2+(2[x]+\{x\})^2}\leq\frac{11}{72}.$$

Nicuşor Zlota, Focşani

L322. Fie $a, b, c \in (0, \infty)$ cu $a+b+c=\sqrt{3}$. Demonstraţi că

$$\frac{\sqrt{3}}{4}\leq\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{3\sqrt{3}}{4}.$$

Generalizare!

Tidor Pricope, elev, Botoşani

L323. Demonstraţi că, pentru oricare numere a, b, c din intervalul $(-1, 1)$, are loc inegalitatea

$$(1-abc)^3(1-a^3)(1-b^3)(1-c^3)\leq(1-a^2b)(1-ab^2)(1-a^2c)(1-ac^2)(1-b^2c)(1-bc^2).$$

Marian Tetiva, Bârlad

L324. Dacă $a, b, c \in (0, \infty)$, demonstraţi inegalităţile:

a) $\sum\frac{2a^2}{b+c}\geq 3\frac{a^2+b^2+c^2}{a+b+c}$;

b) $\sum\frac{a^3}{b^2+c^2}\geq\frac{3}{2}\frac{a^2+b^2+c^2}{a+b+c}$.

Neculai Stanciu, Buzău şi Titu Zvonaru, Comăneşti

L325. Dacă $x, y, z \in (0, \infty)$, demonstraţi că

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq\sqrt{3(x^2+y^2+z^2)}+\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x+y+z}.$$

Leonard Giugiuc, Drobeta-Tr. Severin şi Marian Cucoaneş, Mărăşeşti

Training problems for mathematical contests

A. Junior highschool level

G316. A number of a bags, numbered $1, 2, \ldots, a$, are considered, such that each of them contains b coins, $b > a$. The masses of all the b coins in each bag are expressed by integer numbers and − except one bag − all the coins [coin bags] have the same mass. That bag contains false coins only, each coin having its mass less by p than the mass of a true coin, such that p is not a multiple of $a-1$. Using a scale with digital display, determine the number of the bag with false coins by three weighings only.

Geanina Hăvârneanu, Iaşi

G317. Find the natural numbers n such that $n!+(n+1)!+(n+3)!$ is a perfect cube.

Ioan Viorel Codreanu, Satulung (Maramureş)

G318. Let x, y, z be positive real numbers such that $x\,y\,z = x+y+z+2$. Prove that the following inequalities hold:

$$\sum x \geq 2\sum \frac{1}{\sqrt{x\,y}-1} \geq 6\sum \frac{1}{x\,y-\sqrt{x\,y}+1} \geq 6.$$

Marian Tetiva, Bârlad

G319. If $a, b, c \in (0,1)$, prove that

$$\sum \frac{(a+b)^2}{2} > 3(\sum a^2) - 2(\sum a\,b) + 3(\sum a).$$

Cosmin Manea şi Dragoş Petrică, Piteşti

G320. a) If a, b, c are the side lengths of a triangle, show that

$$0 \leq \frac{|a-b|}{a+b} + \frac{|b-c|}{b+a} + \frac{|c-a|}{c+a} < 2.$$

b) For any $t \in [0,2)$, there exists a triangle whose sides a, b, c have the property that $0 \leq \dfrac{|a-b|}{a+b} + \dfrac{|b-c|}{b+c} + \dfrac{|c-a|}{c+a} \leq t$.

Constantin Dragomir, Piteşti

G321. It is considered, in the interior of the right-angled isosceles triangle ABC, $AB = AC$, the point M such that $m(\widehat{MCA}) = 15^\circ$. If Q is the projection of A on MC and $\dfrac{AQ}{MC} = \dfrac{1}{2}$, determine the measure of the angle $\widehat{MBC}$.

Mihai Berindeanu, Bucureşti

G322. Let $ABCD$ be a convex quadrilateral, $\{O\} = AC \cap BD$, and is the symmetric of O with respect to the midpoint of the segment AD. Prove that $AB \| CD$ if and only if the area of the quadrilateral $APDO$ is the harmonic mean of the areas of triangles ABD, ABC, ACD and BCD.

Claudiu-Ştefan Popa, Iaşi

G323. Let ABC be an arbitrary triangle, D, E, F the midpoints of the sides BC, CA and AB respectively and let M, N, P be the projections of the gravity center on the sides BC, CA and AB respectively. If $\triangle MNP \sim \triangle DEF$, show that the triangle ABC is equilateral.

Temistocle Bîrsan, Iaşi

G324. Let $ABCD$ be an inscribable quadrilateral with $CD = AD + BC$. Prove that the interior angle bisectors of angles $\widehat{A}$ and $\widehat{B}$ intersect each other at a point situated on the side CD.

Neculai Roman, Mirceşti (Iaşi)

G325. It is considered the tetrahedron $ABCD$ and the points M, N, P and Q on the segments AB, BC, CD and AD respectively such that $\dfrac{MA}{MB} = \dfrac{NB}{NC} = \dfrac{PC}{PD} =$

$\dfrac{QD}{QA} = k > 1$. Let $\{X\} = AC \cap MN$, $\{Y\} = BD \cap NP$, $\{Z\} = AC \cap PQ$ and $\{T\} = BD \cap MQ$.

a) Show that the tetrahedrons $ABCD$ and $XYZT$ share the same gravity center.

b) Prove that $\dfrac{\mathcal{V}_{XYZT}}{\mathcal{V}_{ABCD}} = \left(\dfrac{k^2+1}{k^2-1}\right)^2$.

Marius Olteanu, Râmnicu Vâlcea

B. Highschool Level

L316. Let ABC be a right-angled triangle with $m(\widehat{A}) = 90°$ and the points P, Q situated on the smaller side AC, with P between A and Q, such that $m(\widehat{PBA}) = m(\widehat{QBC}) = m(\widehat{C}) \leq 30°$.

a) Show that the centre of the circumcircle of the triangle BCP is the symmetric of point Q with respect to BC.

b) Determine the maximum value of the product $PA \cdot QA$ and deduce that $\tan\alpha \cdot \cot 2\alpha \leq \tan^2\left(\dfrac{\pi}{4} - \dfrac{\alpha}{2}\right)$, $\forall \alpha \in \left(0, \dfrac{\pi}{6}\right]$.

Mihail Frăsilă şi Constantin Petrea, Paşcani

L317. Let O, H be the center of the circumcircle and the orthocenter of the acute-angled triangle ABC. Let us consider the point D on the side AB and point E on the side AC such that A is the centre of the escribed circle to the triangle ODE. Prove that DE is the mid-perpendicular of the line segment AH.

Titu Zvonaru, Comăneşti şi Bogdan Ioniţă, Bucureşti

L318. Let ABC be a right-angled triangle inscribed in the circle $\mathcal{C}$ of radius R and D the foot of the altitude from the vertex of the right angle $\widehat{A}$. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ of centres O_1 and O_2, that are inner-tangent to the circle $\mathcal{C}$ are also tangent to the half-lines $[AB$ & $[AD$, and $[AC$ & $[AD$ respectively. Show that the area $(\triangle AO_1O_2) = \dfrac{8Rr^2}{2R+r}$, where r is the radius of the inscribed circle (incircle) in the triangle ABC.

Neculai Roman, Mirceşti (Iaşi)

L319. Let I be the center of the circle inscribed in triangle ABC. We denote by O_1, O_2 and O_3 the circumcentres of the circumcircles to the triangles BIC, CIA and AIB respectively. Prove that

$$\frac{1}{AB \cdot O_3O_1} + \frac{1}{BC \cdot O_1O_2} + \frac{1}{CA \cdot O_2O_3} \geq \frac{1}{AB \cdot BC} + \frac{1}{BC \cdot CA} + \frac{1}{CA \cdot AB}.$$

Florin Stănescu, Găeşti

L320. In a plane there are given two points B and C, and a straight line Δ which is parallel to BC. A point A is mobile on Δ. Determine the geometric locus of the circumcentre of the median triangle to triangle ABC.

Temistocle Bîrsan, Iaşi

L321. We denote by $[x]$ and $\{x\}$ the integer part and decimal part of the real number x, respectively. Prove that, for any positive real number x, the following inequality holds:

$$\frac{[x]^2}{7[x]^2+(2\{x\}+[x])^2}+\frac{\{x\}^2}{7\{x\}^2+(2[x]+\{x\})^2}\leq\frac{11}{72}.$$

Nicuşor Zlota, Focşani

L322. Let $a,b,c\in(0,\infty)$ with $a+b+c=\sqrt{3}$. Prove that

$$\frac{\sqrt{3}}{4}\leq\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{3\sqrt{3}}{4}.$$

Generalization required !

Tudor Pricope, elev, Botoşani

L323. Prove that, for any three numbers a,b,c in the interval $(-1,1)$, the following inequality holds:

$$(1-a\,b\,c)^3(1-a^3)(1-b^3)(1-c^3)\leq(1-a^2b)(1-a\,b^2)(1-a^2c)(1-a\,c^2)(1-b^2c)(1-b\,c^2).$$

Marian Tetiva, Bârlad

L324. If $a,b,c\in(0,\infty)$, prove the inequalities

a) $\sum\frac{2a^2}{b+c}\geq3\frac{a^2+b^2+c^2}{a+b+c}$;

b) $\sum\frac{a^3}{b^2+c^2}\geq\frac{3}{2}\frac{a^2+b^2+c^2}{a+b+c}$.

Neculai Stanciu, Buzău şi Titu Zvonaru, Comăneşti

L325. If $x,y,z\in(0,\infty)$, prove that

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq\sqrt{3(x^2+y^2+z^2)}+\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x+y+z}.$$

Leonard Giugiuc, Drobeta Tr. Severin şi Marian Cucoaneş, Mărăşeşti

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