L281. Cu notaţiile uzuale în triunghi, demonstraţi că

$$\min\left\{\frac{m_am_b}{h_ah_b}, \frac{m_bm_c}{h_bh_c}, \frac{m_cm_a}{h_ch_a}\right\} \le \frac{R}{2r} \le \max\{\frac{m_am_b}{h_ah_b}, \frac{m_bm_c}{h_bh_c}, \frac{m_cm_a}{h_ch_a}\}.$$

Vasile Jiglău, Arad

L282. Să se arate că, pentru orice $x, y, z > 0$, are loc inegalitatea

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \sqrt[3]{\frac{xyz}{(x+y)(y+z)(z+x)}} \ge 2.$$

Marian Cucoaneş, Mărăşeşti

L283. Arătaţi că $\frac{1}{1-a^4b} + \frac{1}{1-b^4c} + \frac{1}{1-c^4d} + \frac{1}{1-d^4a} \ge \frac{1}{1-a^2bcd} + \frac{1}{1-ab^2cd} + \frac{1}{1-abc^2d} + \frac{1}{1-abcd^2}$, oricare ar fi numerele reale $a, b, c, d \in [0, 1)$.

Marius Olteanu, Râmnicu Vâlcea

L284. Fie a, b numere reale pozitive. Arătaţi că sistemul de ecuaţii $x^2 + y^2 = a^2 + b^2$, $x^3 + y^3 = a^3 + b^3$ are şi alte soluţii reale decât soluţiile evidente (a, b) şi (b, a). Există asemenea soluţii cu ambele componente pozitive?

Marian Tetiva, Bârlad

L285. a) Fie $n \ge 2$ un număr natural şi p cel mai mare divizor prim al lui $n(n+1)$. Fie $\sigma_1, \ldots, \sigma_n$ sumele simetrice fundamentale ale numerelor $1, 2, \ldots, n$. Să se arate că $\sigma_1, \ldots, \sigma_{p-2}$ se divid cu p.

b) Dacă p este un număr prim şi $n \equiv -1 \pmod{p^2}$, atunci $\sigma_1, \ldots, \sigma_{p-2}, \sigma_{p-1}$ se divid cu p.

Marian Tetiva, Bârlad

Training Problems for Mathematical Contests

A. Junior Highschool Level

G276. Determine the maximal value of the real number α such that $\frac{x^3}{a} + \frac{y^3}{b} \ge \frac{\alpha\, x\, y\, (x+y)}{a+b}$, for any positive real numbers x, y, a and b.

Alexandru Blaga, Satu Mare

G277. Let $x_1, x_2, \ldots, x_{2n+1}$ (where $n \in \mathbb{N}^*$) be positive real numbers with $\sum_{i=1}^{2n+1} x_i = 2n+1$. Show that $\sum_{i=1}^{2n+1} \frac{x_i}{n\,x_i^2 + n + 1} \le 1$.

Lucian Tuţescu şi Teodora Rădulescu, Craiova

G278. Prove there are no natural nonzero numbers n so that the number $a_n = 5^n + 5^{n+1} + \ldots + 5^{2n-1}$ is a perfect square.

Radu Miron, elev, Iaşi

G279. Determine the largest natural number n with the property that there are natural numbers $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \ldots + a_n = 5(n-1)$ and $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 2$.

Titu Zvonaru, Comăneşti

G280. Show that there are infinitely many natural numbers n with the property that $n!$ is divisible by $n^3 + n^2 - 36$.

Marian Tetiva, Bârlad

G281. The rectangle $A_1A_2A_3A_4$ has the length $A_1A_2 = L$ and the width $A_2A_3 = l$, where $L, l \in \mathbb{N}^*$, $L > l$ and L is not divisible by l. The rectangles $A_3A_4A_5A_6$, $A_5A_6\,A_7\,A_8$, $\ldots$, $A_{2n-1}A_{2n}\,A_{2n+1}A_{2n+2}$, having the same size as the initial rectangle and $nl > L$ are built. Let us denote by N_1 and P_1 the number and − respectively − the sum of the perimeters of rectangles of length L in the figure thus obtained, and by N_2 and P_2 the number and the sum of the perimeters of rectangles of width L.

a) Does it exist a number $n \geq 4$ such that $N_1 = N_2$?

b) Prove that $P_1 + P_2 < \dfrac{n\,(n+1)(n+5)}{6}(L+l)$.

Cosmin Manea şi Dragoş Petrică, Piteşti

G282. The triangle ABC has $m(\widehat{A}) = 90^\circ$ and $m(\widehat{B}) = 30^\circ$. The symmetric point of A with respect to B is A', the point D is in such a way that $CD \perp BD$, $CD = AB$, while A and D are separated by the side BC. N is the midpoint of BC and $\{M\} = AN \cap A'D$. Prove that $2\,AM = 5\,AC$.

Claudiu-Ştefan Popa, Iaşi

G283. In the convex quadrilateral $ABCD$, with $AC \cap BD = \{O\}$, it is known that $m(\widehat{COD}) = 60^\circ$, $m(\widehat{DAB}) = 110^\circ$ and $AB + CD = AC = BD$. Determine the measures of quadrilateral′s angles.

Dan Nedeianu, Drobeta Tr. Severin

G284. On the circumference of a circle of centre O one considers the points A, B, C such that $m(\widehat{AOB}) = 120^\circ$ and C is the midpoint of the small arc $\widehat{AB}$. For a point T situated on the arc $\widehat{BC}$ which does not contain the point A, let M be a point in the interior of triangle TOA with the property that $m(\widehat{MAT}) = 30^\circ$ and $\widehat{OTM} = 2 \cdot \widehat{OAM}$. Determine the geometric locus of the point M, knowing that the point T runs on the arc $\widehat{BC}$.

Vasile Pravăţ şi Titu Zvonaru, Comăneşti

G285. On the sides of the parallelogram $ABCD$, four half-circles are constructed and the midpoints of these half-circles are respectively denoted by K, L, M, N. Show that:

a) the quadrilateral $KLMN$ is a square;

b) the vertices A, B, C, D are situated on the sides of the square if and only if $ABCD$ is a rectangle.

Temistocle Bîrsan, Iaşi

B. Highschool level

L276. Let ABC be a triangle, I the incenter, and A' the point at which the line AI cuts again the circumcircle of this triangle. Show that the contact points of the tangents from A and A' to the incircle are the vertices of a rectangle if and only if $2\,a = b + c$.

Temistocle Bîrsan, Iaşi

L277. Let us consider the polygon $A_0 A_1 A_2 \dots A_{n+1}$, $n \geq 1$, inscribed in the circle $\mathcal{C}$. Denote by r_i the radii of the circles that are tangent, from the interior, to the circle $\mathcal{C}$ and to the line segments $A_0 A_{i+1}$, $i = \overline{1,n}$ while ρ_i's are the radii of the exterior-tangent circles to circle $\mathcal{C}$ and to the half-lines $(A_0 A_i$, $(A_0 A_{i+1}$, $i = \overline{1,n}$. Let us also denote by r the radius of the circle which is interior-tangent to circle $\mathcal{C}$ and to the segments $A_0 A_1, A_0 A_{n+1}$, while $\rho =$ the radius of the exterior-tangent circle to $\mathcal{C}$ and to the half-lines $(A_0 A_1$, $(A_0 A_{n+1}$.

Show that $\dfrac{r_1}{\rho_1} \cdot \dfrac{r_2}{\rho_1} \cdot \ldots \cdot \dfrac{r_n}{\rho_n} = \dfrac{r}{\rho}$.

Neculai Roman, Mirceşti, Iaşi

L278. Let ABC be a triangle. The perpendicular line at B on AB intersects the perpendicular at C on AC at the point P. Two isogonals drawn from the vertex A cut the lines BP and CP at the points X, respectively Y. If M is the midpoint of the segment XY, show that the triangle MBC is isosceles.

Titu Zvonaru, Comăneşti şi Neculai Stanciu, Buzău

L279. The side bisector AM of the triangle ABC intersects the circle of the nine points asssociated to the triangle at M and N. Prove that $2\,AN < AM$.

Corneliu Mănescu-Avram, Ploieşti

L280. With the usual notations in a triangle, show that − for any natural number $n \geq 2$ − the following inequalities hold:

a) $\sum \sqrt[n]{\dfrac{a}{-a+b+c}} \leq \dfrac{3\,n-4}{n} + \dfrac{2\,R}{n\,r}$; b) $\sum a \cdot \sqrt[n]{\dfrac{a}{-a+b+c}} \leq \dfrac{2\,p\,(R+(n-2)\,r)}{n\,r}$.

Nicuşor Zlota, Focşani şi Corneliu Mănescu-Avram, Ploieşti

L281. With the usual notations in a triangle, show that

$$\min\left\{\frac{m_a\,m_b}{h_a\,h_b}, \frac{m_b\,m_c}{h_b\,h_c}, \frac{m_c\,m_a}{h_c\,h_a}\right\} \leq \frac{R}{2r} \leq \max\left\{\frac{m_a\,m_b}{h_a\,h_b}, \frac{m_b\,m_c}{h_b\,h_c}, \frac{m_c\,m_a}{h_c\,h_a}\right\}.$$

Vasile Jiglău, Arad

L282. Show that, for any x, y, $z > 0$, the following inequality is true:

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \sqrt[3]{\frac{x\,y\,z}{(x+y)(y+z)(z+x)}} \geq 2.$$

Marian Cucoaneş, Mărăşeşti

L283. Show that $\dfrac{1}{1-a^4 b} + \dfrac{1}{1-b^4 c} + \dfrac{1}{1-c^4 d} + \dfrac{1}{1-d^4 a} \geq \dfrac{1}{1-a^2 bcd} + \dfrac{1}{1-ab^2 cd} + \dfrac{1}{1-abc^2 d} + \dfrac{1}{1-abcd^2}$, for any real numbers a, b, c, $d \in [0,1)$.

Marius Olteanu, Râmnicu Vâlcea