L261. Într-un şir de numere reale, un termen se numeşte *acceptabil* dacă poate fi scris ca suma câtorva termeni (nu neapărat distincţi) ai şirului.

a) Dacă toţi termenii şirului sunt numere naturale, cel puţin două relativ prime, atunci toţi termenii şirului, cu excepţia unui număr finit, sunt acceptabili.

b) Daţi exemplu de şir care nu are niciun termen acceptabil.

Radu Miron, elev, Iaşi

L262. Considerăm şirul $(x_n)_{n\geq 0}$ definit prin: $x_0 = 0, x_1 = 1$, $x_2 = 2$, $x_3 = 6$, iar $x_{n+4} = 2x_{n+3} + x_{n+2} - 2x_{n+1} - x_n$. Arătaţi că $\frac{x_1^2}{1^2} + \frac{x_2^2}{2^2} + \ldots + \frac{x_n^2}{n^2} = \frac{x_n x_{n+1}}{n(n+1)}$, $\forall n \in \mathbb{N}^*$.

Constantin Dragomir, Piteşti

L263. Demonstraţi că dacă a, b, c sunt numere reale strict pozitive, atunci este adevărată inegalitatea

$$\frac{8}{9}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) + \frac{abc}{2(a^3+b^3+c^3)} \geq \frac{3}{2}.$$

Titu Zvonaru, Comăneşti

L264. Dacă $A, B \in \mathcal{M}_3(\mathbb{Z})$ sunt două matrice care comută şi $\det(A^2 - AB + B^2) - \det(A^2 + AB + B^2) + 2 = 6\det AB$, arătaţi că $\det(A - B) = 0$.

Florin Stănescu, Găeşti

L265. Fie $a_1, \ldots, a_m$ elementele idempotente din inelul $\mathbb{Z}_n$. Câte dintre sumele $a_i + a_j$, $1 \leq i < j \leq n$, sunt tot elemente idempotente?

Marian Tetiva, Bârlad

Training Problems for Mathematical Contests

A. Junior Highschool Level

G256. Let $p \geq 3$ be a prime number. Show that the number $p \cdot 2^n$, $n \in \mathbb{N}$, can be written in a unique way as a sum (of at least two terms) of consecutive natural numbers.

Elena Iurea, Iaşi

G257. Let $a, n \in \mathbb{N}$, a an odd number and $n \geq 2$. Find the remainder of the division of number a^n by $\frac{a^2+1}{2}$.

Lucian Tuţescu, Craiova şi Dumitru Săvulescu, Bucureşti

G258. Determine the numbers $\overline{abcde}$ (written in base 10), knowing that $a + b + c = e$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{10}{e}$.

Cătălin Calistru, Iaşi

G259. Decompose the number $5^{2015} - 1$ as a product of three factors, each of them greater than 5^{400}.

Constantin Dragomir, Piteşti

G260. Let a, b, c and d be nonzero and distinct natural numbers, so that $abcd$ be a perfect square. Prove that the number $a^4 + b^4 + c^4 + d^4$ can be written as a sume of five nonzero perfect squares.

Dan Nedeianu, Drobeta-Turnu Severin

G261. Determine the numbers $x \in [1, \infty)$ with the property that $[(x-1)^3] = [x-1]^3 = [(x-1)(x-2)(x-3)]$, where $[a]$ is the integer part of the real number a.

Alexandru Blaga, Satu Mare

G262. Let ABC be a triangle, and D, F and N the midpoints of the line segments BC, AD and respectively AC. We consider the points $E \in (AB)$ and $G \in (BD)$, and let us denote $\{M\} = DE \cap FG$. If the points B, M and N are collinear, show that the lines AD and EG are parallel.

Cosmin Manea şi Dragoş Petrică, Piteşti

G263. The triangle ABC is considered with $O-$ the midpoint of side BC, $D-$ the midpoint of the line segment AO, and $BD \cap AC = \{E\}$. Prove that $OE = 2DE$ if and olny if the angle $\widehat{A}$ is a right angle.

Geanina Hăvârneanu, Iaşi

G264. We consider the triangle ABC, D a point situated on its side BC, while E and F are points on the sides AC, respectively AB such that EF is antiparallel to BC. The circles circumscribed to the triangles BDF and CDE intersect, the second time, at the point M. Prove that $ME = MF$ if and only if AD is the bisectrix line of the angle $\widehat{BAC}$.

Bogdan Ioniţă, Bucureşti şi Titu Zvonaru, Comăneşti

G265. We consider the triangle ABC inscribed in the circle $\mathcal{C}$. The circle $\mathcal{C}_1$ is tangent to the circle $\mathcal{C}$ and to the segments AB and BC at the points M, L and respectively K. The straight line AC intersect the circumscribed circles to the triangles AML and CMK at the points R and S, while the points E and F are the midpoints of the arcs $\widehat{RM}$ and $\widehat{SM}$. Show that the points A, C, E and F are concyclic.

Neculai Roman, Mirceşti (Iaşi)

B. Highschool Level

L256. Let M be the midpoint of the side BC of the triangle ABC. Show that the radical axis of the circles of diameters BC and AM passes through the orthocenter of the triangle ABC.

Neculai Roman, Mirceşti (Iaşi)

L257. Let ABC be a triangle in which $a = BC$, $b = CA$, $c = AB$ and $c < a < b$. The Nagel line (determined by the centre of the circumscribed circle and by the gravity centre) intersects the sides AB and AC at the points P, respectively Q. Prove that the lines BQ and CP cut each other on the bisectrix line from A if and only if a is the harmonic mean of the numbers b and c.

Titu Zvonaru, Comăneşti

L258. The circle inscribed in the triangle ABC is tangent to the sides BC, CA and AB at the points D, E and respectively F. The perpendicular line at D on

BC cuts EF at the point Q and the inscribed circle at P. If $\{T\} = BQ \cap AC$ and $\{S\} = CQ \cap AB$, prove that the points S, P and T are collinear.

Bogdan Ioniţă, Bucureşti

L259. We consider the set $A = \{1, 2, 3, \ldots, 10\}$. For any nonempty subset X of A, we define

$$M(X) = \{t \,|\, t = x\,y,\ x \in X,\ y \in Y = A \setminus X\}.$$

Show that $\max\limits_{\emptyset \neq X \subset A} |\, M(X) \,| = 24$.

Gheorghe Iurea, Iaşi

L260. We say that the numbers $a_1, a_2, \ldots, a_n$ form a *united group* if $a_1 + 2\,a_2 + \ldots + n\,a_n = 0$ and $n! \cdot a_1\,a_2 \cdots a_n$ is a perfect square natural number. Show that there exist infinitely many natural numbers n such that there exist united groups with n elements.

Alexandru Blaga, Satu Mare

L261. In a sequence of real numbers, a term is said to be *acceptable* if it can be written as the sum of a couple of terms (not necessarily distinct) of the sequence.

a) If all the terms of the sequence are natural numbers with at least two of them relatively prime, then all terms of the sequence, except a finite number of them, are acceptable.

b) Give an example of a sequence having no acceptable terms.

Radu Miron, highschool student, Iaşi

L262. We consider the sequence $(x_n)_{n \geq 0}$ defined by $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 6$, and $x_{n+4} = 2\,x_{n+3} + x_{n+2} - 2\,x_{n+1} - x_n$. Show that $\dfrac{x_1^2}{1^2} + \dfrac{x_2^2}{2^2} + \ldots + \dfrac{x_n^2}{n^2} = \dfrac{x_{n\,n+1}}{n\,(n+1)}$, $\forall n \in \mathbb{N}^*$.

Constantin Dragomir, Piteşti

L263. Prove that, if a, b, c are strictly positive real numbers, then the following inequality holds:

$$\frac{8}{9}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) + \frac{a\,b\,c}{2(a^3+b^3+c^3)} \geq \frac{3}{2}.$$

Titu Zvonaru, Comăneşti

L264. If $A, B \in \mathcal{M}_3(\mathbb{Z})$ are two commuting matrices and $\det(A^2 - AB + B^2) - \det(A^2 + AB + B^2) + 2 = 6 \det AB$, show that $\det(A - B) = 0$.

Florin Stănescu, Găeşti

L265. Let $a_1, \ldots, a_m$ be the idempotent elements in the ring $\mathbb{Z}_n$. How many of the sums $a_i + a_j$, $1 \leq i < j \leq n$ are idempotent elements, too ?

Marian Tetiva, Bârlad