L244. Fie $a, b, c \in \mathbb{R}_+^*$, $a \geq b \geq c$. Demonstraţi că are loc inegalitatea

$$(a^2+c^2)(ab+ac+bc) - 2ac(a^2+b^2+c^2) \geq 2c(a-b)(a-c)(b-c).$$

Gabriel Dospinescu, Paris şi Marian Tetiva, Bârlad

L245. Fie $f, g : [0,1] \to [0,1]$ două funcţii continue astfel încât $\left| f\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right) \right| \leq \frac{1}{k} f\left(g\left(\frac{k}{n}\right)\right)$, $\forall k, n \in \mathbb{N}^*$, $k \leq n$. Demonstraţi că cele două funcţii sunt egale.

Florin Stănescu, Găeşti

Training problems for mathematical contests

A. Junior highschool level

G236. Determine the natural numbers a, b, c, d and e, strictly larger than 1, with the property that $a+b+c+d+e = abcde - 95$.

Titu Zvonaru, Comăneşti

G237. Prove that n distinct natural numbers $a_1, a_2, \ldots, a_n$ such that the sum $a_1 + a_1 + \ldots + a_n$ is a perfect square and $a_1^2 + a_2^2 + \ldots + a_n^2$ is a perfect cube.

Gheorghe Iurea, Iaşi

G238. The real numbers $x, a_1, a_2, \ldots, a_{100}$ are given. If 51 numbers among $a_1, \frac{a_1+a_2}{2}, \ldots, \frac{a_1+a_2+\ldots+a_{100}}{100}$ are equal to x, prove that at least two among the numbers a_i, $i = \overline{1,100}$, are equal to x.

Cătălin Budeanu, Iaşi

G239. Determine the values of the real number k, knowing that

$$a_1^3 + \ldots + a_{2013}^3 + 4026 \geq k(a_1 + \ldots + a_{2013}), \forall a_i \in [-2, \infty).$$

Lucian Tuţescu, Craiova şi Marian Voinea, Bucureşti

G240. Consider the expression $E(x_1, x_2, \ldots, x_n) = \sum\limits_{i=1}^{n} x_i + 2 \sum\limits_{1 \leq i < j \leq n} x_i x_j$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$, $n \geq 16$, with $\sum\limits_{i=1}^{n} x_i^2 = 1$. Determine the extreme values of this expression.

Petru Asaftei, Iaşi

G241. In the triangle ABC, the concurrent cevian lines AA', BB' and CC' are given, such that $A'B \neq A'C$, and the lines BC aand $B'C'$ meet at point M.

a) Prove that $\frac{2}{MA'} = \left| \frac{1}{A'B} - \frac{1}{A'C} \right|$.

b) Determine the length of the segment MA' as a function of triangles side lengths, in the cases when AA' is an angle-bisector line, respectively the altitude from A.

Neculai Roman, Mirceşti (Iaşi)

G242. In the triangle ABC, the side AB is fixed, and the length of the side AC is constant. Let AA' be the bisector of angle $\widehat{BAC}$, with $A' \in BC$. Show that the perpendicular line at A' on AA' passes through a fixed point.

Claudiu-Ştefan Popa, Iaşi

G243. Let O be the intersection of the diagonals of trapezium $ABCD$, with the big base CD. The points M and N are placed such that AD separates M from O, BC separates N from O, and $\triangle MAD \sim \triangle NCB \sim \triangle OAB$. Show that $BD \cdot AC > AB \cdot MN$.

Cosmin Manea şi Dragoş Petrică, Piteşti

G244. Let $VABC$ be a tetraahedron, and let M, N, P be the mid-points of the edges VA, VB and respectively VC. Prove that

$$2(\mathcal{A}_{MBC} + \mathcal{A}_{NCA} + \mathcal{A}_{PAB}) < \mathcal{A}_{VBC} + \mathcal{A}_{VCA} + \mathcal{A}_{VAB} + 3\mathcal{A}_{ABC}.$$

Mihály Bencze, Braşov

G245. An equilateral triangle has its vertices inside or on the sides of a regular hexagon of side length = 1. The triangle does not contain the centre of the hexagon in its interior. Which is the maximum possible length of triangleş side length?

Marian Tetiva, Bârlad

B. Highschool Level

L236. The points A, B, C are considered on the sphere of centre Ω and radius 7 such that $BC = 4$, $CA = 5$ and $AB = 6$. The perpendicular line on the plane of triangle ABC in the centre of the circle inscribed in this triangle intersects the sphere at the points M and N. Determine the length of the line segment MN.

Temistocle Bîrsan, Iaşi

L237. Let A_1, B_1, C_1 be the mid-points of the sides BC, CA and respectively AB of the acute-angled triangle ABC. The common chord of the circles of diameters BB_1 and CC_1 intersect the line B_1C_1 at A_2; we build in a similar way the points B_2 and C_2. Prove that the lines A_1A_2, B_1B_2 and C_1C_2 are concurrent.

Neculai Roman, Mirceşti (Iaşi)

L238. Let R be a point in the interior of triangle ABC, and $\{M\} = AR \cap BC$, $\{N\} = BR \cap AC$, $\{P\} = CR \cap AB$. We note with α, β and γ the measures of the angles $\widehat{AMB}$, $\widehat{BNC}$, respectively $\widehat{CPA}$. If $\alpha - \beta + \gamma - C + A$, $-\alpha + \beta + \gamma - B + C$ and $\alpha + \beta - \gamma - A + B$ fall in the interval $\left[0, \frac{\pi}{2}\right]$, show that R is the centre of the circle inscribed in the triangle ABC.

Marius Drăgan, Bucureşti

L239. Let $ABCD$ be a circumscrible quadrilateral with $AB \| CD$, while A', B', C' and D' are the tangency (or contact) points of the inscribed circle with the sides AB, BC, CD, respectively DA. Let A'', B'', C'', D'' be the symmetric points of A', B', C', repectively D' with respect to the mid-points of the sides they lie on. Show that:

a) $S_{A'B'C'D'} \leq S_{A''B''C''D''}$;

b) $S_{A'B'C'D'} + S_{A''B''C''D''} \leq S_{ABCD}$.

Marius Olteanu, Rm. Vâlcea

L240. Prove that the following inequality holds in any triangle:

$$\frac{2r}{R}\sum \frac{bc}{ab+ac}+\sum \frac{c}{a+b}\le 3.$$

Marian Tetiva, Bârlad

L241. Show that $\dfrac{\sin^3 x}{(1+\sin^2 x)^2}+\dfrac{\cos^3 x}{(1+\cos^2 x)^2}\le \dfrac{2\sqrt{2}}{9}$, $\forall x\in\mathbb{R}$. (Related to the problem *L211 of RecMat-2/2011.*)

Dumitru Barac, Sibiu

L242. A right-angled parallelepiped has the dimensions (edge lengths) x, y, z and the diagonal length d. Show that

$$\frac{d^4}{ad^4+bx^4}+\frac{d^4}{ad^4+by^4}+\frac{d^4}{ad^4+bz^4}\le \frac{27}{9a+b},$$

any would be $a, b>0$, $6a\ge 5b$. (Related to the problem *L231 of RecMat-2/2012.*)

Titu Zvonaru, Comăneşti

L243. For $m, n\in\mathbb{N}^*$ and $a, b, c\in\mathbb{R}_+^*$, prove the inequality

$$\frac{a^m b^m c^m (a^n+b^n+c^n)^2}{a^{3m+2n}+b^{3m+2n}+c^{3m+2n}}\le 3.$$

(Related to the problem *VIII.149 of RecMat-1/2012.*)

Neculai Stanciu, Buzău

L244. Let a, b, c be positive real numbers with $a\ge b\ge c$. Prove that the following inequality holds:

$$(a^2+c^2)(ab+ac+bc)-2a(a^2+b^2+c^2)\ge 2c(a-b)(a-c)(b-c).$$

Gabriel Dospinescu, Paris şi Marian Tetiva, Bârlad

L245. Let $f, g:[0,1]\to[0,1]$ be two continuous functions with the property that $\left|f\left(\frac{k}{n}\right)-g\left(\frac{k}{n}\right)\right|\le\frac{1}{k}f\left(g\left(\frac{k}{n}\right)\right)$, $\forall k, n\in\mathbb{N}^*$, $k\le n$. Prove that the two functions are equal.

Florin Stănescu, Găeşti
