L132. Pentru a, b, c numere reale pozitive, demonstraţi inegalitatea

$$a\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+c\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\frac{18}{a+b+c}.$$

Florin Stănescu, Găeşti

L133. Dacă a, b, c sunt numere reale pozitive cu $a+b+c=3$, arătaţi că

$$\sum\frac{ab}{ab+a+b}+\frac{1}{9}\sum\frac{(a-b)^2}{ab+a+b}\leq 1.$$

Titu Zvonaru, Comăneşti

L134. Fie n şi k numere întregi pozitive. Demonstraţi identităţile:

a) $\displaystyle\sum_{j=1}^{n-1}\left[\frac{j}{k}\right]=n\left[\frac{n}{k}\right]-\frac{k}{2}\left[\frac{n}{k}\right]\left(\left[\frac{n}{k}\right]+1\right)$; b) $\displaystyle\sum_{j=1}^{n}\left[\sqrt[k]{\frac{n}{j}}\right]=\sum_{q=1}^{[\sqrt[k]{n}]}\left[\frac{n}{q^k}\right]$.

Marian Tetiva, Bârlad

L135. Fie $A, B\in\mathcal{M}_n(\mathbb{R})$ şi X un vector nenul din $\mathbb{R}^n$ astfel încât $AX=O$ şi există $Y\in\mathbb{R}^n$ pentru care $AY=BX$. Notăm cu A_j matricea obţinută înlocuind coloana j a matricei A cu coloana j a matricei B. Arătaţi că $\sum\limits_{j=1}^{n}\det A_j=0$.

Adrian Reisner, Paris

Training problems for mathematical contests

A. Junior highschool level

G216. The numbers from 1 to 9 are arranged on a square with 3×3 cells so that the product of the numbers on row k or column k be a perfect square, for any $k\in\{1,2,3\}$. It is possible to set a odd number in the central cell of the square?

Marius Mâinea, Găeşti

G217. A number of p small squares are drawn on the blackboard. Johnny colours a square, Ann colours three squares, Johnny coulours five of them, Ann colours seven ones and so on. The child who has not sufficient squares to coulour when his/her turn comes on loses. Determine the number p for which the winner of the game is Johnny and establish how many small squares would remain for Ann to colour (as a function of p) .

Gheorghe Iurea, Iaşi

G218. The real numbers $a_1, a_2, \ldots, a_n$ $(n\in\mathbb{N}, n\geq 2)$ are considered. Show that a subset $A\subseteq\{1,2,\ldots,n\}$ exists with the property that $|\sum\limits_{i\in A}a_i|\geq\dfrac{1}{4}\sum\limits_{i=1}^{n}|a_i|$.

Radu Miron, elev, Iaşi

G219. Let a, b, c be nonzero numbers, a odd and $b>c$, such that $a=\dfrac{2bc}{b-c}$ and $(a,b,c)=1$. Show that abc is a perfect square.

Neculai Stanciu, Buzău and Titu Zvonaru, Comăneşti

G220. Determine the digits a with the property that perfect squares of the form $\overline{2\underbrace{aa\ldots a}_{n\text{ time}}6}$ exist.

Adriana Dragomir and Lucian Dragomir, Oţelu-Roşu

G221. Determine the natural number n such that

$$A = 168\left(\frac{1}{[\sqrt{n^2+n+15}]} - \frac{1}{[\sqrt{n^2+n+16}]}\right) \in \mathbb{N}.$$

Mircea Fianu, Bucureşti

G222. Solve the equation $(x+2)^3 = x\,(x^2-2)^5$, $x \in (0,\infty)$.

Dan Nedeianu, Drobeta Tr. Severin

G223. For x, y, $z \geq 0$, show that the following inequality holds.

$$x\,y\,(x^2-y^2)^2 + x\,z\,(x^2-z^2)^2 + y\,z\,(y^2-z^2)^2 \geq 4\,(x-y)^2(x-z)^2(\,y-z)^2.$$

Marian Tetiva, Bârlad

G224. The isosceles trapezium $ABCD$ has the larger base AB and its diagonals are perpendicular at the point O. The parallel line through O to the bases intersects the non-parallel sides at P, respectively R. The point Q is the symmetric point of P with respect to the mid-point of BC. The line RQ intersects AC and BD at the points E, respectively F. Prove that

a) $RQ \perp AD$ and $RQ = AD$;
b) $RE = FQ = CP$ and $PQ = EF$.

Claudiu-Ştefan Popa, Iaşi

G225. Lucian-Georges has a homogeneous triangular plate ABC of mass $= 40$, and a pair of scales. He wants to cut the plate along m straight lines, parallel to BC such that $-$ using these smaller plates as weights placed on the balance pans $-$ he would be able to weigh any object of mass $= n$, with $1 \leq n \leq 40$. How would you advise him to proceed so that m be the minimum possible?

Dan Brânzei, Iaşi

B. Highschool Level

L126. The tangents of the angles of a triangle ABC are rational numbers. Show that the numbers $E_n = \sin^n A \cdot \sin^n B \cdot \sin^n C + \cos^n A \cdot \cos^n B \cdot \cos^n C$ are rational, any would be $n \in \mathbb{N}$.

Cătălin Calistru, Iaşi

L127. The measures of the angles B and C of the triangle ABC are equal to $70°$, respectively $30°$. The points E and F are considered on the side AB such that $\widehat{ACE} \equiv \widehat{ECF} \equiv \widehat{FCB}$. Let AD be the perpendicular from A, $D \in BC$, and $\{M\} = AD \cap CF$. Show that MB is the bisector of angle $\widehat{DMF}$.

Eugeniu Blăjuţ, Bacău

L128. Let ABC be an isosceles triangle with $AB = AC$ and D a point on the side BC. Let us consider the points E and F on the sides AB, respectively AC such that $BD = DE$ and $CD = CF$. Denote $\{T\} = BF \cap CE$. Show that the quadrilateral $BDTE$ can be inscribed in a circle if and only if the quadrilateral $DCFT$ posseses a circumcircle, too.

Titu Zvonaru, Comăneşti

L129. Let be given a triangle ABC and the natural numbers $m \geq n \geq 1$. Build, with a rule and a compass, the points A' in the plane of the triangle such that $A'BC$ has its perimeter and its area m times, respectively n times larger than those of triangle ABC.

Temistocle Bîrsan, Iaşi

L130. a) Let $n \in \mathbb{N}, n \geq 10$. Show that infiunitely many n-tuples $(x_1, x_2, \ldots, x_n)$, with $x_i \in (0,1)$, $\forall i = \overline{1,n}$ and $\sum\limits_{i=1}^{n} x_i = \dfrac{n}{2}$, exist.

b) For an n-tuple as under a), denote $E_n = \dfrac{x_1}{1-x_1} + \dfrac{x_2}{(1-x_1)(1-x_2)} + \ldots + \dfrac{x_n}{(1-x_1)\ldots(1-x_n)}$. Show that $\dfrac{1}{E_n+1}$ can be expressed as a decimal number such that at least $3 \cdot \left[\dfrac{n}{10}\right]$ of its decimal digits are zeros.

Cecilia Deaconescu, Piteşti

L131. Let n be an odd natural number. Establish how many nonzero natural numbers p have the property that $p^2 + n^2$ is a perfect square and determine the largest number with this property.

Marian Panţiruc, Iaşi

L132. For the positive real numbers a, b, c prove the inequality

$$a\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+c\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq \frac{18}{a+b+c}.$$

Florin Stănescu, Găeşti

L133. If a, b, c are positive real numbers with $a+b+c=3$, show that

$$\sum \frac{a\,b}{a\,b+a+b} + \frac{1}{9}\sum \frac{(a-b)^2}{a\,b+a+b} \leq 1.$$

Titu Zvonaru, Comăneşti

L134. Let n and k be positive integer numbers. Prove the identities :

a) $\displaystyle\sum_{j=1}^{n-1}\left[\frac{j}{k}\right] = n\left[\frac{n}{k}\right] - \frac{k}{2}\left[\frac{n}{k}\right]\left(\left[\frac{n}{k}\right]+1\right)$; b) $\displaystyle\sum_{j=1}^{n}\left[\sqrt[k]{\frac{n}{j}}\right] = \sum_{q=1}^{[\sqrt[k]{n}]}\left[\frac{n}{q^k}\right]$.

Marian Tetiva, Bârlad

L135. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ and X a nonzero vector in $\mathbb{R}^n$, such that $AX = 0$ and there exists a $Y \in \mathbb{R}^n$ such that $AY = BX$. We denote by A_j the matrix obtained after replacing column j of matrix A by column j of matrix B. Show that $\sum\limits_{j=1}^{n} \det A_j = 0$.

Adrian Reisner, Paris