L201. Demonstraţi că pentru orice număr prim $p > 2^{2k} + 1$, $\forall k \in \mathbb{N}^*$, numărul $p^{2^{2k+1}} - 1$ se divide cu $2^{2(k+1)} \cdot \prod_{j=1}^{m} q_j$, unde $\{q_j | j = \overline{1, m}\}$ este mulţimea numerelor prime din mulţimea $\{2^i + 1 | i = \overline{1, 2k}\}$.

Neculai Roman, Mirceşti (Iaşi)

L202. Determinaţi numerele reale a şi b pentru care $a\dfrac{\sqrt{n+2}}{n+1} \leq \sqrt{n+2} - \sqrt{n-1} \leq b\dfrac{\sqrt{n+1}}{n}$, $\forall n \in \mathbb{N}^*$.

Gheorghe Iurea, Iaşi

L203. Fie f, g polinoame cu coeficienţii reali, nu ambele constante, iar $P = f + ig \in \mathbb{C}[X]$. Presupunem că rădăcinile lui P sunt numere complexe cu părţile imaginare strict negative. Dacă $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 \neq 0$, arătaţi că rădăcinile polinomului $Q = \lambda f + \mu g$ sunt reale.

Adrian Reisner, Paris

L204. Fie A o matrice pătratică de ordinul n având elementele a_{ij} din mulţimea $\{0, 1\}$ şi următoarele proprietăţi: i) $a_{ii} = 0$ pentru orice $i \in \{1, 2, \ldots, n\}$; ii) dacă $a_{ij} = 1$ (pentru $i \neq j$ din mulţimea $\{1, 2, \ldots, n\}$), atunci $a_{ji} = 0$; iii) pentru fiecare $p \in \{0, 1, \ldots, n-1\}$, matricea are o linie pe care se află exact p elemente egale cu 1.

Să se arate că există o permutare a mulţimii $\{1, 2, \ldots, n\}$ astfel încât, dacă se aplică această permutare liniilor matricei A şi apoi coloanelor matricei astfel obţinute, rezultă în final o matrice cu toate elementele care sunt egale cu 1 situate deasupra diagonalei principale. Care este polinomul caracteristic al unei asemenea matrice?

Marian Tetiva, Bârlad

L205. Calculaţi $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \ln k}{k}$.

Marian Tetiva, Bârlad

Training problems for mathematical contests

A. Junior highschool level

G196. Let M be the set of nonzero natural numbers consisting of even digits only, and having at most 2011 digits. Show that the sum of the inverses of elements in M is less that 4.

Cecilia Deaconescu, Piteşti

G197. Find the highest power of 3 which divides the number $N = 16^{2011} - 2 \cdot 8^{2011} + 3 \cdot 4^{2011} - 2 \cdot 2^{2011} + 1$.

Pedro H.O. Pantoja, Brazil

G198. Solve, in the set of natural numbers, the equation $6^n + 2800 = m^6$.

Andrei Eckstein, Timişoara

G199. Find $b \in \mathbb{N}^*$ such that there exists $a \in \mathbb{N}^*$ with the property that $a^2 + ab + b^2$ is a perfect square.

Gheorghe Iurea, Iaşi

G200. Show that $\frac{a^2}{2b^3+2c^3+5}+\frac{b^2}{2a^3+2c^3+5}+\frac{c^2}{2a^3+2b^3+5}\le\frac{1}{3}$, $\forall a,b,c\in[0,1]$.

Dan Nedeianu, Drobeta Tr. Severin

G201. Let ABC a triangle with $AC\neq BC>AB$. If there exists a point $D\in(BC)$ such $AB^2=BD\cdot BC$ and $AD^2=BD\cdot DC$, show that $-\frac{1}{3}\le\frac{AB^2}{BC^2-AC^2}\le 1$.

Ovidiu Pop, Satu Mare

G202. Let $ABCD$ be a square, and the points M and N situated on the sides AB, respectively AD, such that $AM=DN=k\cdot AB$. Let us denote $\{P\}=CN\cap DM$ and $\{Q\}=AP\cap CD$. Determine the values of the coefficient k for which PQ is an angle-bisector line, respectively a side-bisector in the triangle CDP.

Neculai Roman, Mirceşti (Iaşi)

G203. Let us consider the real numbers a and b with $a<b<2a$. The isosceles triangles ABC şi $A'B'C'$ have the same axis of symmetry, the same support line of their bases and their side lengths are $BC=a$, $AB=AC=b$, $B'C'=b$, $A'B'=A'C'=a$. If $\{M\}=AB\cap A'B'$, $\{N\}=AC\cap A'C'$, show that MN midpoint line in $\triangle ABC$ if and only if $a^3+b^3=2a^2b$.

Temistocle Bîrsan, Iaşi

G204. A point situated on the edge of a tetrahedron is said to be a *bisector point* if it is the intersection point of the angle-bisector lines of two angles of two triangular sides. Show that the number of bisector points of a tetrahedron can be $0,2$ or 6.

Silviu Boga, Iaşi

G205. Two bright panels are situated in (vertical) parallel planes. They have the shapes of two identical rectangles, each of them divided into ten congruent squares by means of a horizontal line and four vertical lines. A bulb is installed at each of the 18 corners of the squares resulting from this division. At a certain moment, two bulbs randomly shine on each panel. Which is the probability that the four shining bulbs lie in the same plane?

Gabriel Popa and Cristian Lazăr, Iaşi

B. Highschool Level

L196. Show that the following inequality holds in any acute-angled triangle:

$$\frac{(\operatorname{ctg}A+\operatorname{ctg}B+\operatorname{ctg}C)^3}{(\operatorname{ctg}A+\operatorname{ctg}B)(\operatorname{ctg}B+\operatorname{ctg}C)(\operatorname{ctg}C+\operatorname{ctg}A)}\ge\frac{8}{3\sqrt{3}}\frac{(1+\cos A\cos B\cos C)^3}{\sin A\sin B\sin C}.$$

Gheorghe Costovici, Iaşi

L197. Let $ABCDEFGH$ be a right-angled parallelepiped, and $\mathcal{S}$ a sphere passing through point A that intersects the line segments AB,AD,AE and AG at M,N,P, respectively Q. Show that $AM\cdot AB+AN\cdot AD+AP\cdot AE=AQ\cdot AG$.

Claudiu Ştefan Popa, Iaşi

L198. Let ABC be an acute-angled triangle inscribed in the circle $\mathcal{C}$. The circle $\mathcal{C}'$ is tangent to the circle $\mathcal{C}$ at point A and also to the side BC at point D. Show that AD is the angle-bisector line of angle $\widehat{BAC}$.

Titu Zvonaru, Comăneşti

L199. Let O be the center of the circumcircle of triangle ABC and $\{M\} = OB \cap AC$, $\{N\} = OC \cap AB$. If $OM = ON$, show that the triangle is either isosceles or right-angled.

Temistocle Bîrsan, Iaşi

L200. With respect to a Cartesian system of coordinates xOy, three points are considered, namely $M\left(\frac{3}{2}, 1\right)$, $B\left(\frac{3a-2}{a}, 0\right)$ and $C\left(0, \frac{2a+3}{a}\right)$, where $a \in \mathbb{R} \setminus \left\{0, \frac{2}{3}, -\frac{3}{2}\right\}$, as well as the family of lines $d_m : y = mx + \frac{2-3m}{2}$, $m \in \mathbb{R}$. Denote by $\mathcal{C}_a$ the circumcircle of the triangle OBC.

a) Prove that, for any $m \in \mathbb{R}$, the line d_m intersects $\mathcal{C}_a$ at two distinct points P and Q.

b) Show that the product $MP \cdot MQ$ is independent of both a and m.

Gabriel Popa and Paul Georgescu, Iaşi

L201. Prove that, for any prime number $p > 2^{2k} + 1$, $\forall k \in \mathbb{N}^*$, the number $p^{2^{2k+1}} - 1$ is divisible by $2^{2(k+1)} \cdot \prod_{j=1}^{m} q_j$, where $\{q_j \,|\, j = \overline{1, m}\}$ is the set of the prime numbers included in the set $\{2^i + 1 \,|\, i = \overline{1, 2k}\}$.

Neculai Roman, Mirceşti (Iaşi)

L202. Find the real numbers a and b such that

$$a\frac{\sqrt{n+2}}{n+1} \leq \sqrt{n+2} - \sqrt{n-1} \leq b\frac{\sqrt{n+1}}{n}, \quad \forall n \in \mathbb{N}^*.$$

Gheorghe Iurea, Iaşi

L203. Let f, g be two polynomials with real coefficients, not both of them constant, and $P = f + ig \in \mathbb{C}[X]$. Assume that the roots of P are complex numbers with strictly negative imaginary parts. If $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 \neq 0$, show that the roots of the polynomial $Q = \lambda f + \mu g$ are real.

Adrian Reisner, Paris

L204. Let A be a square matrix of order n having its entries a_{ij} in the set $\{0, 1\}$, also enjoying the following properties: i) $a_{ii} = 0$ for any $i \in \{1, 2, \dots, n\}$; ii) if $a_{ij} = 1$ (for $i \neq j$ in the set $\{1, 2, \dots, n\}$), then $a_{ji} = 0$; iii) for any $p \in \{0, 1, \dots, n-1\}$, the matrix has a row containing exactly p entries equal to 1.

Show that a permutation of the set $\{1, 2, \dots, n\}$ exists such that, if this permutation is applied to the rows of matrix A and the same permutation is then applied to the matrix just obtained, it is finally obtained a matrix whose all entries equal to 1 are situated above the main diagonal. Which is the characteristic polynomial of such a matrix?

Marian Tetiva, Bârlad

L205. Calculate $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \ln k}{k}$.

Marian Tetiva, Bârlad