Training problems for mathematical contests

A. Junior highschool level

G176. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}_+^*$, with $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_n} = 1$. Prove that

$$\frac{1}{a_1^3 + a_2^2} + \frac{1}{a_2^3 + a_3^2} + \ldots + \frac{1}{a_n^3 + a_1^2} < \frac{1}{2}.$$

Angela Ţigăeru, Suceava

G177. Let $k > 0$ and $a, b, c \in [0, +\infty)$ such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2 + a + k} + \frac{b}{b^2 + b + k} + \frac{c}{c^2 + c + k} \leq \frac{9}{9k + 4}.$$

Titu Zvonaru, Comăneşti

G178. If $n \in \mathbb{N} \backslash \{0, 1\}$, show that $\dfrac{1 - n}{n} \leq \{nx\} - \{x\} \leq \dfrac{n - 1}{n}$, $\forall x \in \mathbb{R}$.

Gheorghe Iurea, Iaşi

G179. Determine the prime numbers a, b, c, d and the number $p \in \mathbb{N}^*$, so that $a^{2^p} + b^{2^p} + c^{2^p} = d^{2^p} + 3$.

Cosmin Manea and Dragoş Petrică, Piteşti

G180. Find the remainder of the division of $S = 2010^{2009!} + 2009^{2008!} + \ldots + 2^{1!} + 1^{0!}$ by 41.

Răzvan Ceucă, hight-school student, Iaşi

G181. Let $k \geq 1$ be a given natural number. Show that there are infinitely many natural numbers n such that n^k divides $n!$.

Marian Tetiva, Bârlad

G182. The triangle ABC is considered with the points $M \in [AB]$, $N \in [BC]$, $P \in [CA]$ such that $MP \| BC$ and $MN \| AC$. Let $\{Q\} = AN \cap MP$ and $\{T\} = BP \cap MN$. Prove that $\mathcal{A}_{AMN} = \mathcal{A}_{PTN} + \mathcal{A}_{QPC}$.

Andrei Răzvan Băleanu, hight-school student, Motru

G183. Let ABC be an isosceles triangle, with $AB = AC$ şi $m(\widehat{A}) < 30°$. Knowing that the points $D \in [AB]$ and $E \in [AC]$ exist such that $AD = DE = EC = BC$, determine the measure of the angle $\widehat{A}$.

Vasile Chiriac, Bacău

G184. The points A_{ij} of coordinates (i, j) are considered in the plane xOy, where $i, j \in \{0, 1, 2, 3, 4\}$. Let $\mathcal{P}$ be the set of the squares with their vertices among the considered points A_{ij}. Find the minimum length of a path consisting of square sides only, which joins the points A_{00} and A_{44}.

Claudiu Ştefan Popa, Iaşi

G185. Show that there exists a coloring of the plane by n colours, where $n \geq 2$ is a given natural number, so that any line segment in phe plane contain points colored by each of the n colours.

Paul Georgescu and Gabriel Popa, Iaşi

B. Highschool Level

L176. Let D, E, F be the projections of the centroid G of the triangle ABC onto the lines BC, CA, and respectively AB. Show that the Cevian lines AD, BE and CF meet at a unique point if and only if the triangle is isosceles.

Temistocle Bîrsan, Iaşi

L177. We consider the triangle ABC with its centroid G and Lemoine′s point L. Denote by M and N the projections of G onto the interior and exterior bisector lines of angle $\widehat{A}$ and let P and Q be the projections of L on the interior and respectively exterior bisector lines of angle $\widehat{A}$. Prove that the lines GK, MN and PQ meet at the same point.

Titu Zvonaru, Comăneşti

L178. Let $ABCD$ be a rhombus with its side length $\ell = 1$ and the points $A_1 \in (AB)$, $B_1 \in (BC)$, $C_1 \in (CD)$, $D_1 \in (DA)$. Prove that $A_1B_1^2 + B_1C_1^2 + C_1D_1^2 + D_1A_1^2 \geq 2\sin^2 A$.

Neculai Roman, Mirceşti, Iaşi

L179. Prove that the following inequality holds in any triangle:

$$\frac{2(9R^2 - p^2)}{9Rr} \geq \frac{\cos^2 A}{\sin B \sin C} + \frac{\cos^2 B}{\sin C \sin A} + \frac{\cos^2 C}{\sin A \sin B} \geq 1.$$

I.V. Maftei and Dorel Băiţan, Bucureşti

L180. Determine the minimum number of factors in the product $P = \sin\frac{\pi}{4^n} \cdot \sin\frac{2\pi}{4^n} \cdot \sin\frac{3\pi}{4^n} \cdot \ldots \cdot \sin\frac{(2^{2n-1}-1)\pi}{4^n}$, $n \in \mathbb{N}^*$, so that $P < 10^{-9}$.

Ionel Tudor, Călugăreni, Giurgiu

L181. Let P be a point on the circular boundary of a half-disc, and d the tangent at P to this boundary. Denote by $\mathcal{C}$ the rotation body obtained by the rotation of the half-disc around the line d. Study the variation of the volume of $\mathcal{C}$ as a function of the position of point P.

Paul Georgescu and Gabriel Popa, Iaşi

L182. Let $(x_n)_{n\geq 1}$ be a sequence of integer numbers with the property that $x_{n+2} - 5x_{n+1} + x_n = 0$, $\forall n \in \mathbb{N}^*$. Show that if a term of the sequence is divisible by 22 then infinitely many terms there of have this property.

Marian Tetiva, Bârlad

L183. Let us consider the numbers $a \in \mathbb{Z}$, $n \in \mathbb{N}^*$ and the polynomial $p(X) = X^2 + aX + 1$. Show that there exist a polynomial with integer coefficients q_n and an integer number b_n such that $p(X)q_n(X) = X^{2n} + b_nX^n + 1$.

Marian Tetiva, Bârlad

L184. Show that the function $f : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$, $f(x, y) = \dfrac{x^2 + y^2 + 2xy - x - 3y + 2}{2}$, is bijective.

Silviu Boga, Iaşi

L185. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property that $|f(x + y) - f(x) - f(y)| \leq |x - y|$, $\forall x, y \in \mathbb{R}$. Show that $\lim_{x\to 0} f(x) = 0$ if and only if $\lim_{x\to 0} xf(x) = 0$.

Adrian Zahariuc, student, Princeton