L162. Dacă $n \in \mathbb{Z}^*$ este fixat, rezolvaţi în $\mathbb{R}$ ecuaţia $\left[\frac{x}{n}\right] = \left[\frac{[x]}{n}\right]$.

Dumitru Mihalache şi Gabi Ghidoveanu, Bârlad

L163. Fie a un număr întreg impar, iar $n \in \mathbb{N}^*$. Arătaţi că polinomul $X^{2^n} + a^{2^n}$ este ireductibil în $\mathbb{Z}[X]$ însă, pentru orice număr prim p, polinomul redus modulo p este reductibil în $\mathbb{Z}_p[X]$.

Dorel Miheţ, Timişoara

L164. O secvenţă $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ de $2n$ numere reale are proprietatea (P) dacă $x_i^2 + y_i^2 = 1$, $\forall i = \overline{1, n}$. Fie $n \in \mathbb{N}^*$ astfel încât pentru orice secvenţă cu proprietatea (P), există $1 \leq i < j \leq n$ cu $x_i x_j + y_i y_j \geq 0,947$. Determinaţi cea mai bună constantă α aşa încât $x_i x_j + y_i y_j \geq \alpha$, pentru orice secvenţă cu proprietatea (P).

Vlad Emanuel, student, Bucureşti

L165. Fie $n \geq 2$ un număr natural. Determinaţi cel mai mare număr natural m pentru care există submulţimile nevide şi distincte $A_1, A_2, \ldots, A_m$ ale lui $A = \{1, 2, \ldots, n\}$, cu proprietatea că fiecare element al lui A este conţinut în cel mult k dintre ele, unde:

a) $k = 2$; b) $k = n$; c) $k = n + 1$.

Marian Tetiva, Bârlad

Training problems for mathematical contests

A. Junior highschool level

G156. If $a, b, c \in \mathbb{R}_+^*$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3$, prove that

$$\frac{a^2+1}{\sqrt{a^2-a+1}} + \frac{b^2+1}{\sqrt{b^2-b+1}} + \frac{c^2+1}{\sqrt{c^2-c+1}} \geq 6.$$

I.V. Maftei, Bucureşti and Mihai Haivas, Iaşi

G157. We say that a natural number has the property (P) if it can be written as the sum of three nonzero perfect squares, and it has the property (Q) if it can be written as the sum of four nonzero perfect squares.

a) Give examples of natural numbers that have: property (P) only; property (Q) only; both property (P) and property (Q).

b) If the numbers $a, b, c \in \mathbb{N}^*$ have an even sum and each of them differs from the sum of the other two numbers, show that $a^2 + b^2 + c^2$ has the property (Q).

Ovidiu Pop, Satu Mare

G158. The equation $x^2 + y^2 + z^2 = (x-y)^2 + (y-x)^2 + (z-x)^2$ with x, y, $z \in \mathbb{N}$ is considered.

a) Show that this equation has infinitely many solutions.

b) If (x, y, z) is a solution to the equation, show that each of the numbers $x\,y$, $y\,z$, $z\,x$ and $x\,y + y\,z + z\,x$ is a perfect square.

Liviu Smarandache, Craiova

G159. Find the last two digits of the number $(70n+6)\cdot 6^{n-1}$.

Ion Săcăleanu, Hârlău

G160. Three sets are considered, namely: $A=\{1,2,3,\ldots,2009\}$; $B=\left\{\frac{a}{a+d}+\frac{b}{b+d}+\frac{c}{c+d}\middle| a,b,c,d\in A \text{ and mutually distinct}\right\}$; $C=\left\{\frac{d}{a+d}+\frac{d}{b+d}+\frac{d}{c+d}\middle| a, b,c,d\in A \text{ and mutually distinct}\right\}$. Determine $A\cap B\cap C$. (*The problem is connected with E: 13650 of Gazeta Matematică 5-6/2008*).

Andrei Crăcană, highschool student, Iaşi

G161. Let M be the set of numbers of the form $\overline{abc}$ with $a\cdot b\cdot c\neq 0$. Determine the maximal cardinal number of a subset N of M such that $x+y\neq 1109$, $\forall x,\, y\in N$.

Petru Asaftei and Gabriel Popa, Iaşi

G162. We may replace the triple of integer numbers (a,b,c) by one of the triples $(2b+2c-a,b,c)$, $(a,2a+2c-b,c)$, $(a,b,2a+2b-c)$. Show that if we start from the triple $(31329,24025,110224)$ and successively aply such replacings only triples consisting of perfect squares are obtained.

Marian Tetiva, Bârlad

G163. Let ABC be a triangle with $m(A)\neq 90^0$ and take the points $B_1\in(AC)$ and $C_1\in(AB)$. Prove that the radical axis of the circles of diameters $[BB_1]$ and $[CC_1]$ passes through the point A if and only if $B_1C_1\parallel BC$.

Neculai Roman, Mirceşti (Iaşi)

G164. Let B,b be given real numbers with $B>b>0$. Among all the circumscriptible trapesiums with the lengths of their bases (respectively) equal to B and b select the one of maximum area.

Claudiu Ştefan Popa, Iaşi

G165. Let ABC be an isosceles triangle ($AB=AC$), M the mid-point of the side $[BC]$, and P a point in the interior of triangle ABM. We denote $\{D\}=BP\cap AC$, $\{E\}=CP\cap AB$. Prove that $BE<CD$ and $PE<PD$.

Cristian Pravăţ, Iaşi and Titu Zvoranu, Comăneşti

B. Highschool level

L156. Let M be a point that is exterior to the circle $\mathcal{C}$ of center O and radius R. We denote by T_1,T_2 the contact points, with this circle, of the tangents from M to $\mathcal{C}$, and let A be the intersection point of the straight line OM with circle $\mathcal{C}$ such that $A\notin[OM]$. Determine the points M with the property that a triangle can be built with the line segments $[MT_1]$, $[MT_2]$ and $[MO]$ as its sides, while a triangle with $[MT_1]$, $[MT_2]$ and $[MA]$ as its sides cannot be built.

Temistocle Bîrsan, Iaşi

L157. In the plane of $\triangle ABC$, we define the transformation $P\to P'$ as follows: 1^0 the point P is projected onto the lines BC, CA, AB at the points D, E and respectively F; 2^0 the symmetric points of D, E, F with respect to the mid-points of the sides $[BC]$, $[CA]$, and respectively $[AB]$ are denoted as D', E', F'; 3^0 P' is the common (or meeting) point of the perpendicular lines at D', E', F' on BC, CA and respectively AB. Show that the transformation $P \to P'$ coincides with the symmetry with respect to O − the center of the circumscribed circle to $\triangle ABC$.

Temistocle Bîrsan, Iaşi

L158. We consider the point T in the interior of triangle ABC with its side $[BC]$ fixed and its vertex A mobile such that $\widehat{ATB} = \widehat{BTC} = \widehat{CTA}$. Determine the position of the point A in the plane of the triangle such that $m(\widehat{BAC}) = \alpha < \dfrac{5\pi}{6}$, and the sum of the distances from T to the vertices of the triangle is maximum.

Cătălin Calistru, Iaşi

L159. If $a, b, c \in \mathbb{R}_+^*$ and $x \in \left(0, \dfrac{\pi}{2}\right)$, prove the inequality

$$a\left(\frac{\sin x}{x}\right)^3 + b\left(\frac{\sin x}{x}\right)^2 + c\left(\frac{\sin x}{x}\right) + 3\sqrt[3]{abc}\left(\frac{\tan x}{x}\right) \geq 6 \cdot \sqrt[3]{abc}.$$

D.M. Bătineţu-Giurgiu, Bucureşti

L160. Prove that the following inequality holds in any triangle:

$$m_a + m_b + m_c \geq 6\,r\left(\frac{m_a}{m_b + m_c} + \frac{m_b}{m_a + m_c} + \frac{m_c}{m_a + m_b}\right) \geq 9\,r.$$

Marius Olteanu, Rm. Vâlcea

L161. If $a, b, c \in \mathbb{R}_+^*$ and $a + b + c = 1$, prove the inequality

$$3 + \sum \frac{(a-b)^2 + (a-c)^2}{1+a} \leq 4(a^2 + b^2 + c^2)\left(\sum \frac{1}{1+a}\right).$$

Titu Zvoranu, Comăneşti

L162. If $n \in \mathbb{Z}^*$ is fixed, solve in $\mathbb{R}$ the equation $\left[\dfrac{x}{n}\right] = \left[\dfrac{[x]}{n}\right]$.

Dumitru Mihalache and Gabi Ghidoveanu, Bârlad

L163. Let a be an odd number and let n be a nonzero number. Prove that the polynomial $X^{2^n} + a^{2^n}$ is irreducible in $\mathbb{Z}[X]$, while it factors modulo f for all prime p.

Dorel Miheţ, Timişoara

L164. A sequence $x_1, x_2, \ldots, x_n$, y_1, $y_2, \ldots$, y_n of $2n$ real numbers is said to have the property (P) if $x_i^2 + y_i^2 = 1$, $\forall i = \overline{1,n}$. Let $n \in \mathbb{N}^*$ such that for any sequence with property (P) there are subscripts i, j with $1 \leq i < j \leq n$ such that $x_i\, x_j + y_i\, y_j \geq 0.947$. Determine the best constant α such that $x_i\, x_j + y_i\, y_j \geq \alpha$, for any sequence with property (P).

Vlad Emanuel, student, Bucureşti

L165. Let $n \geq 2$ be a natural number. Determine the largest natural number m such that m nonempty and distinct subsets $A_1, A_2, \ldots, A_m$ of $A = \{1, 2, \ldots, n\}$ exist with the property that each element of A belongs to at most k such subsets, where:

a) $k = 2$; *b*) $k = n$; *c*) $k = n + 1$.

Marian Tetiva, Bârlad