$$\frac{x+2}{2x^2+1}+\frac{y+2}{2y^2+1}+\frac{z+2}{2z^2+1}\geq 3.$$

Titu Zvonaru, Comăneşti şi Nela Ciceu, Bacău

L142. Considerăm $n\in\mathbb{N}^*$, numerele reale strict pozitive $a_1<a_2<\cdots<a_n$ şi A mulţimea tuturor sumelor $\pm a_1\pm a_2\pm\cdots\pm a_n$, unde semnele se aleg în toate modurile posibile. Arătaţi că $|A|>\frac{n^2+n+2}{2}$ şi determinaţi numerele a_n pentru care se atinge egalitatea.

Gheorghe Iurea, Iaşi

L143. Să se arate că pentru p număr natural prim şi $m,n\in\{0,1,\ldots,p-1\}$, $m>n$, avem $\binom{2p+m}{2p+n}\equiv 2\binom{p+m}{p+n}-\binom{m}{n}\pmod{p^2}$.

Marian Tetiva, Bârlad

L144. Fie $p\in\mathbb{N}$, $p\geq 2$; definim şirurile $(x_n)_{n\geq 1}$ şi $(y_n)_{n\geq 1}$ prin: $x_1=\sqrt{p(p-1)}$, $x_{n+1}=\sqrt{p(p-1)+x_n}$, $y_n=\{2^np^{n-1}x_n\}$, $\forall n\in\mathbb{N}^*$, unde $\{\cdot\}$ desemnează partea fracţionară. Să se arate că şirul (y_n) este strict monoton.

Sorin Puşpană, Craiova

L145. Fie $0<\alpha<\beta$; definim şirurile $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ prin $x_0=\alpha$, $y_0=0$, $x_{n+1}=\int\limits_{x_n}^{y_n}e^{-\frac{\alpha^2}{t^2}}dt$, $y_{n+1}=\int\limits_{y_n}^{x_n}e^{-\frac{\beta^2}{t^2}}dt$, $\forall n\in\mathbb{N}$. Arătaţi că cele două şiruri sunt convergente şi aflaţi limitele lor.

Marius Apetrei, Iaşi

Training problems for mathematical contests

A. Junior highschool level

G136. Determine the real numbers x, y, z satisfying the equation

$$2^{-x}+3\cdot 2^{-y}+2^{-z}=2^x+3\cdot 2^{y+2}+2^{z+2}=9.$$

Andrei Nedelcu, Iaşi

G137. Let us consider $a,b,c\in\mathbb{Q}_+^*$ and $\lambda=\frac{2\sqrt{a}+\sqrt{b}-\sqrt{c}}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$. It is required to express, in terms of a, b, c and λ, the real number $\mu=\frac{\sqrt{a}-\sqrt{b}+\sqrt{c}}{2\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

I. V. Maftei, Bucureşti and Mihai Haivas, Iaşi

G138. *a*) The positive real numbers a, b, c satisfy the equation $4abc=a+b+c+1$. Prove that $\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{b^2+a^2}{c}\geq 2(ab+bc+ca)$.

b) The positive real numbers a, b, c satisfy the inequality $\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{b^2+a^2}{c}\leq 2(ab+bc+ca)$. Show that $a+b+c+1\leq 4abc$.

Andrei Laurenţiu Ciupan, highschool student, Bucureşti

G139. Denise writes down on the blackboard the numbers 1, 2, 3, ..., 2008. She chooses two numbers, deletes them from the blackboard and replaces them by the modulus of their difference, repeating this operation until a single number remains on the blackboard. Can Denise proceed in such a way that the remaining number be 2007? What about 2008?

Julieta Grigoraş, Iaşi

G140. A polygon with n sides is divided into $n-2$ triangles by means of $n-3$ of its diagonals that do not cut each other(s) at interior points (such a partition being called a *triangulation* of the polygon). Let us denote by T_0 the number of triangles whose sides are all diagonals of the polygon, and by T_2 the number of triangles with two of their sides being sides of the polygon as well, the third side being a diagonal of the polygon. Prove that $T_2 = T_0 + 2$.

Marian Tetiva, Bârlad

G141. It is considered a network of straight lines that form congruent squares by intersecting themselves. We mark $2n+1$ corners of such squares, $n \geq 2$, so that any line in the network pass through at most one marked vertex point. Show that at least two marked points exist such that they are separated, both along horizontal and vertical directions, by an odd number of lines in the network.

Petru Asaftei, Iaşi

G142. We say that the vertex A of the triangle ABC has the property (P) if $AX < BC$, $\forall X \in (BC)$. Show that if each vertex of $\triangle ABC$ enjoys the property (P) then the triangle is equilateral.

Doru Buzac, Iaşi

G143. We consider the triangle ABC with D, D' two points on the line BC such that $\widehat{CAD} \equiv \widehat{ABC}$, iar $\widehat{BAD'} \equiv \widehat{ACB}$. The interior bisectrices of the angles $\widehat{BAD}$ and $\widehat{CAD'}$ cut the line BC at E, respectively F. Show that the circle circumscribed to $\triangle AEF$ and the circle inscribed in $\triangle ABC$ are concentric.

Neculai Roman, Mirceşti (Iaşi)

G144. Let $ABCD$ be a quadrilateral with $AB = BC$. Show that $m(\widehat{BAD}) + m(\widehat{BCD}) = 90^\circ$ if and only if $AB^2 \cdot CD^2 + AD^2 \cdot BC^2 = AC^2 \cdot BD^2$.

Ioan Săcăleanu, Hârlău

G145. It is considered the isosceles triangle ABC with $AB = AC$ and a point M is taken on the open arc $\overset{\frown}{BC}$, which does not contain the corner A, of the circle circumscribed to the triangle. Show that

$$\sqrt{MB \cdot MC} < MA < \sqrt{MB \cdot MC} + \frac{AB \cdot AC}{\sqrt{MB \cdot MC}}.$$

Gheorghe Costovici, Iaşi

B. Highschool level

L136. Let A, B, C be three points on the sphere $\mathcal{S}$ of center O, while M_1 and M_2 are two exterior points with respect to the sphere S such that OM_1 and OM_2 intersect the plane (ABC) at two points that are interior to $\triangle ABC$. If $M_1A \geq M_2A$, $M_1B \geq M_2B$ and $M_1C \geq M_2C$, show that $M_1O \geq M_2O$.

Cătălin Ţigăeru, Suceava

L137. We consider $\triangle ABC$ inscribed in the circle $\mathcal{C}$ and let $\mathcal{C}_1$ be the circle of center O_1, which is tangent to AB, BC and to the circle $\mathcal{C}$ at M, K, and respectively L. The parallel line through B to MK intersects the lines LM and LK at R, respectively S. Show that the angle $\widehat{RO_1S}$ is acute.

Neculai Roman, Mirceşti (Iaşi)

L138. Let ABC be a triangle with $AB \neq AC$, $m(\widehat{A}) < 90°$, the angle $\widehat{A}$ being the largest angle of this triangle. We denote by M the midpoint of $[BC]$ and by T the intersection point of the simedian from A with the mid-perpendicular of $[BC]$. Show that $2\,AM < AT$.

Titu Zvonaru, Comăneşti and Cristian Pravăţ, Iaşi

L139. Fie $A_1A_2\cdots A_n$ be a regular polygon, and M a variable point inside the polygon or on its sides. Determine the highest value of the product $f(M) = MA_1 \cdot MA_2 \cdot \cdots \cdot MA_n$, as well as the points M at which this maximum is reached, in each of the cases: *a)* $n = 3$; *b)* $n = 6$.

Dumitru Mihalache and Marian Tetiva, Bârlad

L140. Let $a, b, c \in \mathbb{R}_+^*$ be numbers such that $(a+b)^2 + (b+c)^2 + (c+a)^2 + (a+b)(b+c)(c+a) = 4$. Prove that $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}$.

Andrei Vrăjitoarea, highschool student, Craiova

L141. If x, y, z are real reale positive numbers with $x^3 + y^3 + z^3 = 3$, then

$$\frac{x+2}{2x^2+1} + \frac{y+2}{2y^2+1} + \frac{z+2}{2z^2+1} \geq 3.$$

Titu Zvonaru, Comăneşti and Nela Ciceu, Bacău

L142. We consider $n \in \mathbb{N}^*$, the strictly positive real numbers $a_1 < a_2 < \cdots < a_n$ and $A =$ the set of all the sums $\pm a_1 \pm a_2 \pm \cdots \pm a_n$, where the signs are chosen in all possible ways. Show that $|A| > \frac{n^2+n+2}{2}$ and determine the numbers a_n for which the equality is achieved.

Gheorghe Iurea, Iaşi

L143. Show that, for a prime natural number p and $m, n \in \{0, 1, \ldots, p-1\}$, $m > n$, we have $\binom{2p+m}{2p+n} \equiv 2\binom{p+m}{p+n} - \binom{m}{n} \pmod{p^2}$.

Marian Tetiva, Bârlad

L144. Let $p \in \mathbb{N}$, $p \geq 2$; we define the sequences $(x_n)_{n\geq 1}$ şi $(y_n)_{n\geq 1}$ by: $x_1 = \sqrt{p(p-1)}$, $x_{n+1} = \sqrt{p(p-1) + x_n}$, $y_n = \left\{2^n p^{n-1} x_n\right\}$, $\forall n \in \mathbb{N}^*$, where $\{\cdot\}$ denotes the decimal (or fractional) part. Show that the sequence (y_n) is strictly monotone.

Sorin Puşpană, Craiova

L145. Let $0 < \alpha < \beta$; we define the sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ by $x_0 = \alpha$, $y_0 = 0$, $x_{n+1} = \int_{x_n}^{y_n} e^{-\frac{\alpha^2}{t^2}}dt$, $y_{n+1} = \int_{y_n}^{x_n} e^{-\frac{\beta^2}{t^2}}dt$, $\forall n \in \mathbb{N}$. Show that the two sequences are convergent and find their limits.

Marius Apetrei, Iaşi