L121. Fie $n \in \mathbb{N}^*$ dat. Să se arate că există câţiva termeni ai şirului $\left(\frac{1}{m^3}\right)_{m \geq n+1}$ a căror sumă este mai mare decât $\frac{1}{(n+1)(2n+1)}$.

Dumitru Mihalache şi Marian Tetiva, Bârlad

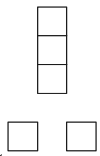
L122. La un campionat de fotbal participă 2^n echipe, astfel încât dintre oricare două se poate dinainte indica echipa mai bună. În prima etapă, echipele se împart aleator în perechi şi dispută câte un meci, echipa mai bună trecând în etapa următoare. Procedeul se repetă până la finală.

a) Care este probabilitatea ca a doua echipă ca valoare să iasă vicecampioană?

b) Dacă se dispută şi o finală mică, ce probabilitate este ca, în plus, cea de-a treia echipă ca valoare să se claseze pe locul 3?

Irina Mustaţă, studentă, Bremen

L123. Pe o tablă 8×9 se aşează dreptunghiuri 3×1 şi "figuri" de forma unui dreptunghi 1×3 căruia îi lipseşte pătratul median (ca în desenul alăturat). "Figurile" şi dreptunghiurile nu se pot roti şi nu au puncte interioare comune. Să se arate că există o mulţime S de 18 pătrate 1×1 astfel încât, dacă pe tablă rămân 2 pătrate neacoperite de dreptunghiuri sau "figuri", atunci cele două pătrate sunt obligatoriu din S.



Gabriel Dospinescu, student, Paris

L124. Fie $n \in \mathbb{N}^*$ fixat.Determinaţi matricele $A \in \mathcal{M}_n(\mathbb{C})$ pentru care ${}^t\left(\overline{A}\right) \cdot A = I_n$, iar $A^{2007} + A + I_n = O_n$ (cu $\overline{\cdot}$ am notat operaţia de conjugare).

Vlad Emanuel, elev, Sibiu

L125. Fie $f : \mathbb{R} \to \mathbb{R}$ o funcţie periodică şi lipschitziană (există $L > 0$ pentru care $|f(x) - f(y)| \leq L|x - y|$, $\forall x, y \in \mathbb{R}$), iae $(x_n)_{n \geq 1}$ un şir strict crescător, cu $\lim\limits_{n \to \infty} x_n = +\infty$ şi $\lim\limits_{n \to \infty} (x_{n+1} - x_n) = 0$. Să se arate că mulţimea punctelor limită ale şirului $(f(x_n))_{n \geq 1}$ coincide cu $\operatorname{Im} f$.

Paul Georgescu şi Gabriel Popa, Iaşi

Training problems for mathematical contests

A. Junior highschool level

G116. Find all the natural numbers N of four distinct nonzero digits with the property that the diferrence between the largest number obtained by permuting the four digitis of N and the smallest number obtained in the same manner equals just N.

Maria Miheţ, Timişoara

G117. Let us consider the set $A = \{1, 2, 3, \ldots, 98\}$. Show that two elements exist among any 50 elements arbitrarily chosen from A such that their sum is a perfect cube.

Titu Zvonaru, Comăneşti

G118. 2007 points are considered in the interior of a parallelogram with its acute angle equal to 30° and the lengths of its sides of 17 cm and 59 cm. Show that three among these points can be selected so that the area of the triangle determined by them be at most equal to $\frac{1}{4}\,\text{cm}^2$.

Mihai Haivas, Iaşi

G119. Let $n \in \mathbb{N}^*$ and $A = \{\varepsilon_0 2^0 + \cdots + \varepsilon_n 2^n \mid \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}\}$, and $B = \{m \mid m \in 2\mathbb{Z} + 1, |m| \leq 2^{n+1} - 1\}$. Show that $A = B$.

Dorel Miheţ, Timişoara

G120. Solve in $\mathbb{N}$ the equation $x!(y!)^{2005} = (z!)^{2007}$.

Anca Ştefania Tuţescu, high-school student, Craiova

G121. For $0 < a, b < 3/2$, prove that the inequality

$$\sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b+3}} + \frac{1}{\sqrt{a} + \sqrt{b}} \leq \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

holds.

Andrei Laurenţiu Ciupan, high-school student, Bucureşti

G122. Let G be the gravity center of $\Delta\, ABC$ and G' its projection on the line BC. Show that $G' \notin [BC]$ if and only if $3a^2 < \left|b^2 - c^2\right|$.

Temistocle Bîrsan, Iaşi

G123. Let ΔABC be an equilateral triangle. Show that any point M in the plane with the property that $MB = MA + MC$ can be determined using a tracing square only. (A tracing square can be used for tracing straight lines and right angles).

Nicolae Ivăşchescu, Craiova

G124. Let ABC be a triangle with the midpoint of $[BC]$ denoted A' and P, Q the projections of A' on AB and respectively AC. Prove that $4PQ \leq AB + BC + CA$.

Adrian Zahariuc, high-school student, Bacău

G125. Let $ABCD$ be a square, $M \in (AB)$, $\{O\} = AC \cap BD$, $\{S\} = CM \cap DA$, and $\{E\} = SO \cap MD$. We consider $AA' \perp (ABC)$, $AA' = AB$, I the midpoint of $[A'D]$ and $\{H\} = MI \cap A'E$. Show that:

a) $MD \perp (A'AE)$; *b)* $\dfrac{V_{A'ADH}}{V_{MADH}} = \left(\dfrac{AB}{AM}\right)^2$.

Petru Răducanu, Iaşi

B. Highschool level

L116. The circle inscribed in ΔABC is tangent to the side BC at the point D_1, and the $A-$ exinscribed circle is tangent to the same side at point D_2. The straight line AD_2 intersects the inscribed circle at the points S and T. Show that ΔSTD_1 is a right-angled triangle.

Titu Zvonaru, Comăneşti

L117. Let us consider ΔABC, $D \in (BC)$, and $\mathcal{C}_1$, $\mathcal{C}_2$ the exinscribed circles to the triangles ADB and ADC that are tangent to BC. Show that a common tangent that passes between the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ also passes through the contact point with BC of the $A-$ exinscribed circle to ΔABC.

Neculai Roman, Mirceşti, Iaşi

L118. Let M be a point of the ellipse $\mathcal{E}$ whose foci are F, F'. The straight lines MF and MF' intersect the ellipse at points A, respectively A'. Show that, when the point M runs over $\mathcal{E}$, the line AA' is ceaselessly tangent to a fixed curve that is required to be determined.

Adrian Reisner, Paris

L119. Let $n \in \mathbb{N}$ and $a, b, c \in \mathbb{R}_+^*$ with $ab + bc + ca = 3$. Show that

$$a^{n+3} + b^{n+3} + c^{n+3} + 2abc(a^n + b^n + c^n) \geq 9.$$

Titu Zvonaru, Comăneşti and Bogdan Ioniţă, Bucureşti

L120. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove the inequality

$$\left(\frac{a_1}{a_2}\right)^{(n-1)^2} + \left(\frac{a_2}{a_3}\right)^{(n-1)^2} + \cdots + \left(\frac{a_n}{a_1}\right)^{(n-1)^2} \geq \frac{a_1 a_2^{2n-1} + a_2 a_3^{2n-1} + \cdots + a_n a_1^{2n-1}}{a_1^2 a_2^2 \cdots a_n^2}.$$

Marian Tetiva, Bârlad

L121. Let $n \in \mathbb{N}^*$ be given. Show that a couple of terms of the sequence $\left(\frac{1}{m^3}\right)_{m \geq n-1}$ exist whose sum is greater than $\frac{1}{(n+1)(2n+1)}$.

Dumitru Mihalache and Marian Tetiva, Bârlad

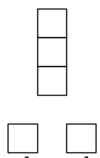
L122. A football championship is attended by 2^n teams such that the best team among each pair of teams can be predicted. In the first stage, the teams are randomly grouped in pairs and play by one match each pair so that the better team passes to the next stage. The procedure is repeated until the final match has to be played.

a) Which is the probability for the second team as to its value to be classified as vice-champion?

b) If a small final match is also disputed, which is the probability for the third team as to its value to get classified on the third position?

Irina Mustaţă, student, Bremen

L123. On a table of size 8×9, rectangles of size 3×1 are placed together with "figures" with the shape of a 1×3 rectangle with the middle square missing (as in the drawing aside). The "figures" and the rectagles cannot be rotated and they have no common interior points. Show that there exists a set S of 18 squares 1×1 such that, if only two squares remain on the table that are not covered by rectangles or "figures", then the two squares are necessarily in S.



Gabriel Dospinescu, student, Paris

L124. Let $n \in \mathbb{N}^*$ be fixed. Determine the matrices $A \in \mathcal{M}_n(\mathbb{C})$ such that ${}^t(\overline{A}) \cdot A = I_n$, and $A^{2007} + A + I_n = O_n$ ($\overline{\cdot}$ denotes the conjugate of the element under the bar).

Vlad Emanuel, high-school student, Sibiu

L125. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodical and lipschitzian function (i.e., there exists $L > 0$ such that $|f(x) - f(y)| \leq L \cdot |x - y|, \forall x, y \in \mathbb{R}$), and let $(x_n)_{n \geq 1}$ be a strictly increasing sequence with $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$. Show that the set of limit points of the sequence $(f(x_n))_{n \geq 1}$ is $\operatorname{Im} f$.

Paul Georgescu and Gabriel Popa, Iaşi