**L101.** Fie $a, n \geq 2$ două numere întregi. Să se arate că $\prod_{k=0}^{n-1} \frac{a^n - a^k}{n-k} \in \mathbb{Z}$.

**Adrian Zahariuc, elev, Bacău**

**L102.** Fie $p = 2k+1$ un număr prim. Atunci

$$S_1 = \sum_{i=k+1}^{2k} C_{p+i-1}^{i} \equiv 2^p - 2 \left(\bmod p^2\right), \quad S_2 = \sum_{i=1}^{k} C_{p+i-1}^{i} \equiv 2 - 2^p \left(\bmod p^2\right).$$

**Marius Pachiţariu, elev, Iaşi**

**L103.** Fie $a$, $b$, $c$, $d$ reale astfel încât $\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right)\left(1+d^2\right) = 16$. Arătaţi că

$$-3 \leq ab + bc + cd + da + ac + bd - abcd \leq 5.$$

Mai mult, avem egalitate în cel puţin una din inegalităţile de mai sus dacă şi numai dacă $a + b + c + d = abc + bcd + cda + dab$.

**Gabriel Dospinescu, student, Paris**

**L104.** Fie $x_0 > 0$ şi $x_n = x_{\left[\frac{n}{2}\right]} + x_{\left[\frac{n}{3}\right]} + \frac{n}{6}$, pentru orice $n > 0$.

*a*) Să se arate că şirul $\left(\frac{x_n}{n}\right)_{n \geq 1}$ este convergent la 1.

*b*) Să se arate că dacă $\alpha > \log_3 \frac{5}{2}$, atunci $\lim_{n \to \infty} \frac{x_n - n}{n^\alpha} = 0$.

**Gabriel Dospinescu, student, Paris**

**L105.** Să se determine toate funcţiile continue $f : (0, \infty) \to \mathbb{R}$, care verifică ecuaţia funcţională

$$n x^{n-1} f\left(x^n\right) = (x+1) f(x), \quad \forall x \in (0, \infty),$$

unde $n \in \mathbb{N}^*$, $n$ fixat.

**Marian Tetiva şi Dumitru Mihalache, Bârlad**

# Training problems for mathematical contests

## A. Junior highschool level

**G96.** Let $a = x^{12m} + x^{12n}$, with $m, n \in \mathbb{N}^*$. Prove that $a$ is divisible by 13 if and only if $x$ is divisible by 13.

**Artur Bălăucă, Botoşani**

**G97.** Find $a, b \in \{0, 1, 2, \ldots, 9\}$, $a \neq 0$, such that $A = \overline{a\underbrace{bb\ldots b}_{n \text{ times}}}$, $n > 2$ is a perfect square.

**Gheorghe Iurea, Iaşi**

**G98.** Find $m, n \in \mathbb{N}^*$ such that $\frac{m}{n} + \frac{n+1}{m^2} \in \mathbb{N}^*$.

**Gabriel Dospinescu, student, Paris**

**G99.** Let $m$, $n$ two positive integer such that $m$ divides $n-1$. All positive integers between 1 and $n$ are put on a circle in an arbitrary way. One compute the sum of any set of $m$ neighbors numbers. Prove that among all these sums, there are two of them for which their difference is strictly grater than $m-1$.

**Titu Zvonaru, Comăneşti**

**G100.** In how many ways could one colour a square $3 \times 3$ such that in any little square $2 \times 2$ to be four different colours?

**Gabriel Popa, Iaşi**

**G101.** Prove the following inequality

$$4\left(\frac{1}{a(1+bc)^2}+\frac{1}{b(1+ca)^2}+\frac{1}{c(1+ab)^2}\right) \leq 1+\frac{16}{(1+bc)(1+ca)(1+ab)},$$

$\forall a, b, c \in (0, \infty)$ under the condition $abc = 1$. When does the equality holds?

**Gabriel Mîrşanu and Andrei Nedelcu, Iaşi**

**G102.** Find the maximal value of the parameter $m \in \mathbb{R}_{+}^{*}$ such that

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{a^2+b^2}{c} \geq m\sqrt{3\left(a^2+b^2+c^2\right)}, \quad \forall a, b, c \in \mathbb{R}_{+}^{*}.$$

**Dorel Băiţan and I. V. Maftei, Bucureşti**

**G103.** For $a, b, c \in (0, 1)$ and $a + b + c = 2$, show that

$$abc \geq 8(1-a)(1-b)(1-c).$$

**Alexandru Negrescu, highschool student, Botoşani**

**G104.** The triangle $ABC$ has $m(\widehat{BAC}) = 120°$. Let $O \in (BC)$ such that $[AO$ is the angle bisector of $\widehat{BAC}$. Let $D$ be a point on $[AO$ such that $[BC$ to be interior angle bisector of $\widehat{ABD}$. Prove that $AD + BD = AB + AC$ and $AB + AC \geq 4\,AO$.

**Petru Răducanu, Iaşi**

**G105.** Let $ABCD$ be a trapezium with $AB$, $CD$ ($AB > CD$) as the bases and consider that the diagonals of the trapezium intersect in $O$. We construct $MN$ to be the mean line of the trapezium and the parallel $PQ$, through $O$, to the bases of the trapezium ($M, P \in (AB)$, $N, Q \in (BC)$ ). Prove that the trapeziums $ABMN$ and $PQCD$ have the diagonal respectively parallel.

**Claudiu-Ştefan Popa, Iaşi**

## B. Highschool level

**L96.** Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}$ such that $\mathcal{C}_1$ and $\mathcal{C}_2$ touch each other externally in $D$ and each of them is interior tangent to $\mathcal{C}$ in $B$ and $C$, respectively. The common interior tangent to the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ cuts the circle $\mathcal{C}$ in $A$ and $A_1$. The line $AB$ cuts the circle $\mathcal{C}_1$ in $K$ and the line $AC$ cuts the circle $\mathcal{C}_2$ in $L$. From the point $M$ laying on the circle $\mathcal{C}$ we construct the tangent lines $MT_1$ and $MT_2$ to the circles $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively ($T_1 \in \mathcal{C}_1$, $T_2 \in \mathcal{C}_2$). If $MA < MA_1$ prove that $MT_1 + MT_2 = \dfrac{A_1M}{A_1D} \cdot KL$ and $|MT_1 - MT_2| = \dfrac{AM}{AD} \cdot KL$.

**Neculai Roman, Mirceşti (Iaşi)**

**L97.** Prove that, in any triangle, the following inequality holds

$$\frac{1}{m_a^2(m_b+m_c-m_a)}+\frac{1}{m_b^2(m_c+m_a-m_b)^2}+\frac{1}{m_c^2(m_a+m_b-m_c)^2} \geq \frac{1}{S^2}.$$

**I. V. Maftei and Dorel Băiţan, Bucureşti**

**L98.** Let $ABC$ be an arbitrary triangle. Prove that

1) $\sin^4 A + \sin^4 B + \sin^4 C \geq \frac{27}{2}\left(\frac{r}{R}\right)^3$;

2) $\cos^4 A + \cos^4 B + \cos^4 C \geq \frac{3}{8}\left(\frac{r}{R}\right)^4\left(5 - \frac{r_a}{r}\right)\left(5 - \frac{r_b}{r}\right)\left(5 - \frac{r_c}{r}\right)$,

where $R$ is the radius of the circumcircle circle, $r$ is the radius of the inscribed circle and $r_a$, $r_b$, $r_c$ are the radius of the exinscribed circles.

**Oleg Faynshteyn, Leipzig, Germany**

**L99.** *a*) Which is the minimal number of points of integer coordinates contained in a plane, such that no mater how they are chosen, there exist three of them with their centre of gravity expressed by integer coordinates.

*b*) Prove that, in an $n$-dimensional space exist $2^{n+1}$ points of integer coordinates such that any 3 of them have the center of gravity with at least one coordinate not an integer.

**Irina Mustaţă, student, Bremen, Germany**

**L100.** Let $x \in (0,1)$; prove that there exist $n \in \mathbb{N}^*$ such that $\{nx\} \in \left[\frac{1}{3}, \frac{2}{3}\right)$. (By $\{\cdot\}$ we denoted the fractional part.)

**Ciprian Baghiu and Gheorghe Iurea, Iaşi**

**L101.** Let $a, n \geq 2$ two integers. Prove that $\prod_{k=0}^{n-1} \frac{a^n - a^k}{n-k} \in \mathbb{Z}$.

**Adrian Zahariuc, highschool student, Bacău**

**L102.** Let $p = 2k+1$ be a prime number. Then

$$S_1 = \sum_{i=k+1}^{2k} C_{p+i-1}^i \equiv 2^p - 2 \left(\operatorname{mod} p^2\right), \quad S_2 = \sum_{i=1}^{k} C_{p+i-1}^i \equiv 2 - 2^p \left(\operatorname{mod} p^2\right).$$

**Marius Pachiţariu, highschool student, Iaşi**

**L103.** Let $a$, $b$, $c$, $d$ some real numbers such that

$$\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right)\left(1+d^2\right) = 16.$$

Prove that

$$-3 \leq ab + bc + cd + da + ac + bd - abcd \leq 5.$$

Moreover, at least one from the above inequalities becomes an equality if and only if $a + b + c + d = abc + bcd + cda + dab$.

**Gabriel Dospinescu, student, Paris**

**L104.** Let $x_0 > 0$ and $x_n = x_{\left[\frac{n}{2}\right]} + x_{\left[\frac{n}{3}\right]} + \frac{n}{6}$, for any $n > 0$.

*a*) Prove that the sequence $\left(\frac{x_n}{n}\right)_{n\geq 1}$ is convergent to 1.

*b*) Prove that if $\alpha > \log_3 \frac{5}{2}$, then $\lim\limits_{n\to\infty} \frac{x_n - n}{n^\alpha} = 0$.

**Gabriel Dospinescu, student, Paris**

**L105.** Find all continuous functions $f : (0,\infty) \to \mathbb{R}$, which verify the functional equation

$$nx^{n-1} f\left(x^n\right) = (x+1) f(x), \quad \forall x \in (0,\infty),$$

where $n \in \mathbb{N}^*$, $n$ is fixed.

**Marian Tetiva and Dumitru Mihalache, Bârlad**