tiv Q.

Paul Georgescu şi Gabriel Popa, Iaşi

L83. Să se calculeze

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{\frac{1}{2}}+\left(1+\frac{1}{n}\right)^{\frac{2}{3}}+\cdots+\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}-n\right].$$

Marius Olteanu, Râmnicu Vâlcea

L84. Fie $n\in\mathbb{N}$, $n\geq 3$ şi

$$A=\Big\{x>0;\quad x=a_0+a_1\sqrt[n]{n}+\cdots+a_{n-1}\sqrt[n]{n^{n-1}};$$
$$a_0,a_1,\ldots,a_n\in\mathbb{Z};\quad n-1\mid a_0+a_1+\cdots+a_n\Big\}.$$

Determinaţi $\inf A$.

Paul Georgescu şi Gabriel Popa, Iaşi

L85. Fie $f:\mathbb{R}\to\mathbb{R}$ o funcţie pentru care mulţimea punctelor în care f are limită finită la stânga este densă în $\mathbb{R}$. Să se arate că mulţimea punctelor în care f este continuă este de asemenea densă în $\mathbb{R}$. (O mulţime $D\subset\mathbb{R}$ se numeşte *densă* în $\mathbb{R}$ dacă orice interval deschis al axei reale conţine măcar un element din D.)

Gabriel Dospinescu, Paris, şi Marian Tetiva, Bârlad

Training problems for mathematical contests

A. Junior high school level

G76. Solve the system $\begin{cases} x^2-y=u^2\\ y^2-z=v^2\\ z^2-x=t^2\end{cases}$ in the set of natural numbers.

Adrian Zanoschi, Iaşi

G77. *i*) Prove that $\dfrac{a^2}{a-b}+\dfrac{b^2}{b-c}>a+2b+c$ for any $a,b,c\in\mathbb{R}$, $a>b>c$.

ii) Prove that $\dfrac{a^2-b^2}{c}+\dfrac{c^2-b^2}{a}+\dfrac{a^2-c^2}{b}\geq 3a-4b+c$ for any $a,b,c\in\mathbb{R}$, $a\geq b\geq c>0$.

Ioan Şerdean, Orăştie

G78. Prove that

$$\frac{b(a+c)}{c(a+b)}+\frac{c(b+d)}{d(b+c)}+\frac{d(a+c)}{a(d+c)}+\frac{a(b+d)}{b(a+d)}\geq 4$$

for any $a,b,c,d\in(0,\infty)$.

Artur Bălăucă, Botoşani

G79. Prove that

$$xy+yz+zx\geq 3+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}$$

for any $x,y,z\in(0,\infty)$ such that $x+y+z=xyz$.

Florina Cârlan and Marian Tetiva, Bârlad

G80. Let A be the set of all sums of type $\pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2n+1)^2$, $n \in \mathbb{N}$, for any combination of the signs. Prove that $A = \mathbb{Z}$. (*Regarding Erdös-Surányi theorem.*)

Petru Asaftei, Iaşi

G81. Let $n \in \mathbb{N}^*$ and $k \in \{0, 1, \ldots, 2^n - 1\}$. Prove that there exists a set $A \subset \mathbb{R}$ with n elements which has precisely k subsets whose sum of elements are strictly positive.

Adrian Zahariuc, high school student, Bacău

G82. Find minimal number of moves required to transfer a knight on a chessboard from the square $A1$ to the square $H8$. For this minimal number, find the number of distinct paths of minimal length.

Gheorghe Crăciun, Plopeni, and Gabriel Popa, Iaşi

G83. Let $ABCD$ be a convex quadrilateral and let $M, N \in (AB)$, $P, R \in (CD)$ such that $AD \cap BC \cap MR \cap NP \neq \emptyset$. Prove that $\frac{BM}{MN} \cdot \frac{NA}{DP} \cdot \frac{PR}{RC} \cdot \frac{CD}{AB} = 1$.

Andrei-Sorin Cozma, junior high school student, Iaşi

G84. Let $ABCD$ be a trapezoid with $AB \| CD$, $AB < CD$. Consider $E \in (AD)$ and $F \in (BC)$ such that $\frac{AE}{ED} = \frac{CF}{FB}$. The line EF meets BD and AC in M, respectively in N. Prove that $\frac{MN}{EF} = \frac{DC - AB}{DC + AB}$.

Andrei Nedelcu, Iaşi

G85. Let ABC be a given triangle and let A', B', C' be the legs of its bisectors. Let D, E be points on the side (BC) such that $D \in (BE)$ and the cevians AD and AE are isogonal. Prove that DB' and EC' intersect each other in a point situated on AA'. (*Regarding Proposition 1, p. 99, RecMat - 2/2004.*)

Titu Zvonaru, Comăneşti

B. High school level

L76. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be circles which are internally tangent to a given circle $\mathcal{C}$ in M, respectively in $N, M \neq N$. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are secant or externally tangent and that the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$ meets $\mathcal{C}$ in A and B. Let us denote by K, respectively by L, the points in which the lines AM, AN meet again $\mathcal{C}_1$, respectively $\mathcal{C}_2$. Prove that $AB \geq 2KL$ and characterize the case of equality.

Neculai Roman, Mirceşti (Iaşi)

L77. Let P_1, P_2, $\ldots$, P_{13} points with integer coordinates in a plane such that any given three are not collinear. Prove that there exists at least a triangle $P_iP_jP_k$ such that its centroid has integer coordinates.

Vasile Pravăţ and Titu Zvonaru, Comăneşti (Bacău)

L78. Consider $(P_n)_{n \in \mathbb{N}}$ a sequence of points on the unit circle such that $m(\widehat{P_nOP_{n+1}}) = \operatorname{arctg} \frac{5}{12}$ for all $n \in \mathbb{N}$, $\widehat{P_nOP_{n+1}}$ being considered as an oriented angle. Prove that for any point P on the unit circle there exists $j \in \mathbb{N}$ such that

$P_j \in \operatorname{Int} \mathcal{C}\left(P, \frac{1}{2005}\right)$.

Lucian Lăduncă and Andrei Nedelcu, Iaşi

L79. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be such that $a_1 + a_2 + \cdots + a_n = 0$ and $\max\{|a_i - a_j|; 1 \le i < j \le n\} \le 1$. Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 \le \frac{1}{n}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ and characterize the case of equality.

Marius Pachiţariu, high school student, Iaşi

L80. Consider an alphabet with four letters a, b, c, d. Using this alphabet, one can construct words according to the following rules: b cannot succeed a, c cannot succeed b, d cannot succeed c and a cannot succeed d. How many palyndromes of length n, $n \ge 2$, can one construct in this manner? (By *palyndrome* we mean a word in which the k-th letter coincides with the $n-k+1$-th letter for all $k \in \{1, 2, \ldots, n\}$.)

Irina Mustaţă, high school student, Iaşi

L81. Let $n \ge 1$ be a fixed natural number. An infinite chessboard is colored in black and white in the usual manner. A set C of squares is then called *connected set* if one can reach any square in C starting from any given square in C through a succession of moves in C from a square to a neighboring square (with a common edge).

Let S be a connected set with $4n$ squares. One calls the *chromatic index* of S the quotient between the number of white squares and the number of black squares. Find the maximal and the minimal value of the chromatic index.

Adrian Zahariuc, high school student, Bacău

L82. Find $P, Q \in \mathbb{R}[X]$ such that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \{p(x) + \sin q(x)\}$ is periodic where $p, q : \mathbb{R} \to \mathbb{R}$ are the polynomial functions associated to P, respectively to Q.

Paul Georgescu and Gabriel Popa, Iaşi

L83. Find

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{\frac{1}{2}} + \left(1+\frac{1}{n}\right)^{\frac{2}{3}} + \cdots + \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} - n\right].$$

Marius Olteanu, Râmnicu Vâlcea

L84. Let $n \in \mathbb{N}$, $n \ge 3$ and

$$A = \Big\{x > 0;\ \ x = a_0 + a_1\sqrt[n]{n} + \cdots + a_{n-1}\sqrt[n]{n^{n-1}};$$
$$a_0, a_1, \ldots, a_n \in \mathbb{Z};\ \ n-1 \mid a_0 + a_1 + \cdots + a_n\Big\}.$$

Find $\inf A$.

Paul Georgescu and Gabriel Popa, Iaşi

L85. Let $f : \mathbb{R} \to \mathbb{R}$ a function such that the set of points in which f has finite left-sided limit is dense in $\mathbb{R}$. Prove that the set of continuity points of f is also dense in $\mathbb{R}$. (A subset D of $\mathbb{R}$ is called *dense* in $\mathbb{R}$ if any open interval of $\mathbb{R}$ contains at least an element of D.)

Gabriel Dospinescu, Paris, and Marian Tetiva, Bârlad