# O generalizare a identităţii Botez - Catalan

***Ioana OLAN***[1]

În 1872, **N. Şt. Botez** publică o lucrare originală în care apare identitatea

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n+1}=1-\frac{1}{2}\left(\frac{1}{1\cdot 3}+\frac{1}{2\cdot 5}+\cdots+\frac{1}{n\left(2n+1\right)}\right),\quad n\in\mathbb{N}^*,$$

care, dacă ţinem seama de formula de descompunere

$$\frac{1}{2k\left(2k+1\right)}=\frac{1}{2k}-\frac{1}{2k+1},\quad k\in\mathbb{N}^*,$$

se aduce la forma

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n},\quad n\in\mathbb{N}^*,\qquad (1)$$

numită *identitatea Botez - Catalan.* Ne propunem să-i dăm o generalizare.

Amintim o demonstraţie a formulei (1), generalizarea obţinându-se în acelaşi fel:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=$$

$$=\left(1+\frac{1}{2}-2\cdot\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}-2\cdot\frac{1}{4}\right)+\cdots+\left(\frac{1}{2n-1}+\frac{1}{2n}-2\cdot\frac{1}{2n}\right)=$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)=$$

$$=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

**Propoziţie.** *Pentru* $n\in\mathbb{N}^*$ *şi* $m\in\mathbb{N}$*, are loc egalitatea*

$$1-\frac{2^m-1}{2^m}+\frac{1}{3^m}-\frac{2^m-1}{4^m}+\cdots+\frac{1}{\left(2n-1\right)^m}-\frac{2^m-1}{\left(2n\right)^m}=\frac{1}{\left(n+1\right)^m}+\cdots+\frac{1}{\left(2n\right)^m}.\qquad (2)$$

(Pentru $m=1$ se obţine identitatea (1).)

**Demonstraţie.** Într-adevăr, avem:

$$1-\frac{2^m-1}{2^m}+\frac{1}{3^m}-\frac{2^m-1}{4^m}+\cdots+\frac{1}{\left(2n-1\right)^m}-\frac{2^{m-1}}{\left(2n\right)^m}=$$

$$=\left(1+\frac{1}{2^m}-2^m\frac{1}{2^m}\right)+\left(\frac{1}{3^m}+\frac{1}{4^m}-2^m\frac{1}{4^m}\right)+\cdots+\left(\frac{1}{\left(2n-1\right)^m}+\frac{1}{\left(2n\right)^m}-2^m\frac{1}{\left(2n\right)^m}\right)=$$

$$=1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\cdots+\frac{1}{\left(2n-1\right)^m}+\frac{1}{\left(2n\right)^m}-2^m\left(\frac{1}{2^m}+\frac{1}{4^m}+\cdots+\frac{1}{\left(2n\right)^m}\right)=$$

$$=1+\frac{1}{2^m}+\frac{1}{3^m}+\cdots+\frac{1}{\left(2n-1\right)^m}+\frac{1}{\left(2n\right)^m}-\left(1+\frac{1}{2^m}+\cdots+\frac{1}{n^m}\right)=$$

$$=\frac{1}{\left(n+1\right)^m}+\frac{1}{\left(n+2\right)^m}+\cdots+\frac{1}{\left(2n\right)^m},\quad \text{q.e.d.}$$

**Cazuri particulare.** Pentru $m=2$ şi $m=n$, formula (2) devine:

$$1-\frac{3}{2^2}+\frac{1}{3^2}-\frac{3}{4^2}+\cdots+\frac{1}{\left(2n-1\right)^2}-\frac{3}{\left(2n\right)^2}=\frac{1}{\left(n+1\right)^2}+\frac{1}{\left(n+2\right)^2}+\cdots+\frac{1}{\left(2n\right)^2},$$

$$1-\frac{2^n-1}{2^n}+\frac{1}{3^n}-\frac{2^n-1}{4^n}+\cdots+\frac{1}{\left(2n-1\right)^n}-\frac{2^n-1}{\left(2n\right)^n}=\frac{1}{\left(n+1\right)^n}+\frac{1}{\left(n+2\right)^n}+\cdots+\frac{1}{\left(2n\right)^n}.$$

[1] Elevă, cl. a VIII-a, Colegiul Naţional "C. Negruzzi", Iaşi