Two Determinants

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Abstract. In this short Note, a sufficient condition for the equality of two determinants is given.
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Let A, B, C and D be four real square matrices in $\mathrm{Mat}(n, \mathbf{R})$. Under what conditions is it true that

$$\det\left(\begin{bmatrix} A & C \\ B & D \end{bmatrix}\right) = \det(AD - BC), \tag{1}$$

where $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ is a block matrix from $\mathrm{Mat}(2n, \mathbf{R})$?

In a Matrix Analysis class taught by author, many students believed, incorrectly, that (1) was true in general, by analogy with its $n = 1$ counterpart. In fact, for $n = 2$, $A = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives a simple counterexample.

Below we indicate a nice sufficient condition under which (1) holds true. There seem to be no simple necessary and sufficient descriptions of (1) for arbitrary matrices.

Proposition. Let A, B, C and D be four real square matrices in $\mathrm{Mat}(n, \mathbf{R})$. If A and B commute then (1) holds true.

Proof. Assume first that A is invertible. Then the following LU-decomposition holds true:

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & D - BA^{-1}C \end{bmatrix}, \tag{2}$$

where I and 0 are the identity, respectively zero, $n \times n$ matrices. Then

$$\begin{aligned} \det\left(\begin{bmatrix} A & C \\ B & D \end{bmatrix}\right) &= \det\left(\begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & D - BA^{-1}C \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix}\right) \det\left(\begin{bmatrix} A & C \\ 0 & D - BA^{-1}C \end{bmatrix}\right) \\ &= \det A \det(D - BA^{-1}C)) = \det(AD - ABA^{-1}C) \\ &= \det(AD - BAA^{-1}C) = \det(AD - BC). \end{aligned} \tag{3}$$

In (3) we have used, besides the hypothesis, the simple fact that for upper/lower triangular block matrices we have

$$\det\left(\begin{bmatrix} P & Q \\ 0 & S \end{bmatrix}\right) = \det\left(\begin{bmatrix} P & 0 \\ R & S \end{bmatrix}\right) = \det P \det S, \tag{4}$$

even if the square matrices P and S have different sizes.

If A is not invertible, in the above argument we replace A by $A + \epsilon I$, $\epsilon > 0$ small enough, and get the desired result by a continuity argument, letting $\epsilon \to 0$.

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