Food for Thought

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Abstract. In this Note, an interesting counting problem - given at a problem solving contest in a North Texas college - is presented, solved, and discussed.

Keywords: counting problem, inductive proof, uniqueness.

MSC 2010: 97D40.

The following Discrete Math problem, given at a problem solving contest in a North Texas college, attracted a lot of interest among the students and teachers of local high schools and colleges. It is now also presented for the benefit and enjoyment of the Romanian high school math enthusiast, who is urged to try to solve it before reading the solution shown here.

Problem. *While discussing math n people seated at a round table eat a combined total of $n-1$ slices of pizza. Show that there is a unique way of counting the people around the table so that first person eats no pizza, the first two people eat no more than a slice, the first three people eat no more than two slices, etc.*

Solution. As the statement suggests, counting is done circularly around the table. One can start at any location, then proceed in one direction, say clockwise, and count progressively. There are n such possible countings. We have to show existence and uniqueness for the required counting property.

Given a sequence of $n \geq 1$ real numbers, $x_1, x_2, \ldots, x_n$, we will say that it has the property (N) if

$$(N) \qquad x_1 + x_2 + \cdots + x_k \leq k-1, \quad k = 1, 2, \ldots, n.$$

The solution to the existence part will be accomplished by induction on n.

The case $n = 1$ is obvious. Assume now that for $n-1$ people, $n \geq 2$, a counting with property (N) always exists. To the end of proving that one also exists for n persons, start with any counting $1, 2, \ldots, n$ of them, and denote by $x_1, x_2, \ldots, x_n$ the number of slices of pizza eaten by the respective persons. The x's are non-negative integers, and by hypothesis $x_1 + x_2 + \cdots + x_n = n-1$. Clearly there must be some i, $1 \leq i \leq n$ with $x_i = 0$, and $x_{i+1} \neq 0$. (If $i = n$, take $i+1$ to be 1). Identify then persons i and $i+1$ (say, they are husband and wife!) and assign to this super-person the job of eating $x_i + x_{i+1} - 1$ slices of pizza. Leave all the other people alone, as location and pizza eating habits go. We have now $n-1$ "people" eating together a combined total of $n-2$ slices of pizza. By the inductive hypothesis, they can be circularly counted such that property (N) is accommodated. Without loss of generality, we can assume that the original counting of the n persons is such that for the $n-1$ people created, $x_1, x_2, \ldots, x_{i-1}, x_i + x_{i+1} - 1, x_{i+2}, \ldots, x_n$ is already a sequence having property (N). The claim then is that $x_1, x_2, \ldots, x_n$ has property (N).

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If $1 \le k \le i-1$, property (N) is clear. Also, $x_1 + x_2 + \cdots + x_{i-1} + x_i = x_1 + x_2 + \cdots + x_{i-1} \le i-2 < i-1$, and $x_1 + x_2 + \cdots + x_i + x_{i+1} = x_1 + x_2 + \cdots + (x_i + x_{i+1} - 1) + 1 \le (i-1) + 1 = i$. Finally, for $k \ge i+2$, $x_1 + x_2 + \cdots + x_i + x_{i+1} + \cdots + x_k = x_1 + x_2 + \cdots + (x_i + x_{i+1} - 1 + 1) + \cdots + x_k \le (k-2) + 1 = k-1$. The inductive proof is complete.

For uniqueness, assume $x_1, x_2, \ldots, x_n$ and $x_j, \ldots, x_n, x_1, \ldots, x_{j-1}$ both have property (N), for some $j > 1$. Then $x_1 + x_2 + \cdots + x_{j-1} \le j-2$ from the first counting, and $x_j + \cdots + x_n \le (n-j+1) - 1 = n-j$, from the second. Thus, $x_1 + x_2 + \cdots + x_n \le (j-2) + (n-j) = n-2$, a contradiction.

Note. If countings other than circular are allowed, the uniqueness is lost. Think about four people eating three slices, distributed as $0, 0, 1, 2$. Then $0, 0, 2, 1$ and $0, 1, 0, 2$ are also solutions.

In fact, for such countings the proof of the existence part is a lot simpler: *Counting people increasingly from frugal to gourmand has property* (N).

Recreaţii ... matematice

I. *Descifraţi mesajul adresat colaboratorilor de către revistă în aritmogriful următor:*

10	9	12	1	13	5	7								
10	9	8	10	9	7	6	1	1						
3	7	5	9	3	7	5	1	8	9					
3	4	11	6	4	3	9	14	5	9					
5	4	5	4	10	2	10								
8	2	11	7	15	2	10	7	5	2	10	1	11	2	10

II. *Ce informaţie transmite numărul* $11111100000_{(2)}$?

(*Răspuns la pag. 94*)

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