Multiplicative duality in triangles

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Abstract. Using the general duality principle in the geometry of the triangle, the authors suggests how it is possible to prove both well-known and new formulas.

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Duality is one of the most fruitful principles in mathematics. Simply said, duality means that every thing has two sides, or every problem can be considers from two different sides. This has many advantages: Mostly the difficulties of the two sides are different. Thus, we can choose the easier way to solve the problem. If you have found an interesting feature of an object its dual property is often interesting, too.

An important example of duality in mathematics is governed by the product of two different values. Looking at a product as the product of two real numbers, due to commutativity it is identical which of the two numbers is the first, which is the second. But often the values – if they describe real things – have different meanings. In physics this is well know. Any value has a unit, e.g., power is voltage times current.

In Euclidean geometry such an example is the area, the product of two lengths. Although, they have the same units, being perpendicular to each other, they point in different directions. This is especially interesting for the triangle, because there is no way to point out two fixed directions.

1. The area of a triangle as a dual product sizes. We consider a given triangle $\triangle ABC$ with side lengths a, b, c and area $S_{\triangle ABC}$. This area can be calculate as the half of the product of one side and the corresponding altitude. We denote the altitudes by h_A, h_B and h_C. To forget about the half in the product, we denote the half of the side lengths by an index 2: $a_2 = a/2$, $b_2 = b/2$, $c_2 = c/2$. Now, we have

$$S_{\triangle ABC} = a_2 h_A = b_2 h_B = c_2 h_C. \tag{2}$$

If we have two variables x and y and the area of the triangle is $S = xy$, we call the variables x and y dual to each other. If we choose the length unit in such a way, that the area of our triangle is 1, then $x = \frac{1}{y}$. If we have a formula or some other relationship with different variables of type x, then we get a similar relationship for the dual variables y, by replacing the x-values by $\frac{1}{y}$.

2. Heron's area formula for the altitudes. Given the sides of a triangle, then we can calculate the area by Heron's formula:

$$S = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

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We introduce new variables (in the sequel this turns out to be very useful):

$$\begin{aligned} p &= a_2+b_2+c_2=\frac{a+b+c}{2}, \\ p_A &= -a_2+b_2+c_2=\frac{-a+b+c}{2}, \\ p_B &= a_2-b_2+c_2=\frac{a-b+c}{2}, \\ p_C &= a_2+b_2-c_2=\frac{a+b-c}{2}; \end{aligned}$$

then Heron′s formula is equivalent to

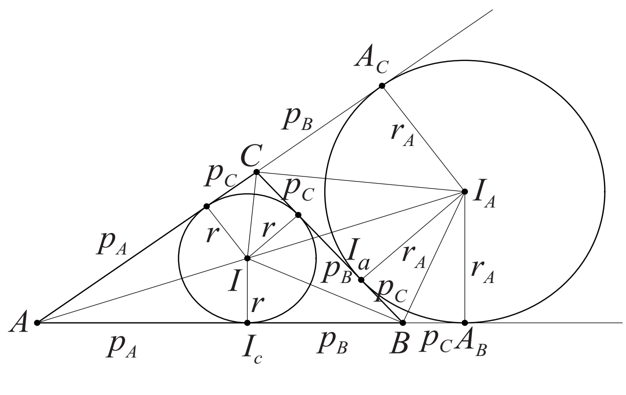
$$\begin{aligned} (3) \qquad S^2 &= p\cdot p_A\cdot p_B\cdot p_C= \\ &= (a_2+b_2+c_2)(-a_2+b_2+c_2)(a_2-b_2+c_2)(a_2+b_2-c_2). \end{aligned}$$

Here, p is the semiperimeter (one half of the triangleş perimeter). The values p_A, p_B and p_C are parts of the side lengths, defined by the touch point of the incircle (see the picture and the text below).

If we replace in (2), the half sides by their dual quantities, the altitudes, i.e., $a_2 \Leftrightarrow \frac{1}{h_A}$, $b_2 \Leftrightarrow \frac{1}{h_B}$, $c_2 \Leftrightarrow \frac{1}{h_C}$, we obtain a formula to calculate the area of the triangle for given altitudes:

(4)

$$\frac{1}{S^2}=\left(\frac{1}{h_A}+\frac{1}{h_B}+\frac{1}{h_C}\right)\left(-\frac{1}{h_A}+\frac{1}{h_B}+\frac{1}{h_C}\right)\left(\frac{1}{h_A}-\frac{1}{h_B}+\frac{1}{h_C}\right)\left(\frac{1}{h_A}+\frac{1}{h_B}-\frac{1}{h_C}\right).$$



3. The area, the incircle and the escribed circles. Besides formula (2), there are other lengths in a triangle, giving the area as their product. For this purpose we draw in a triangle the inscribed circle with radius r and escribed circles (touching the sides from outer) with radii r_A, r_B and r_C.

The values of the side partitions p_A, p_B and p_C are easily calculated from the system of equations

$$(5) \qquad a=2a_2=p_B+p_C,\ b=2b_2=p_C+p_A,\ c=2c_2=p_A+p_B\ .$$

In addition, the formulas

$$p = p_A + p_B + p_C, \tag{6}$$

and

$$p_A = p - 2a_2,\ p_B = p - 2b_2,\ p_C = p - 2c_2 \tag{7}$$

are easy to check, too.

Now, the area can be calculated by

$$S_{\triangle ABC} = S_{\triangle ABI} + S_{\triangle BCI} + S_{\triangle CAI} = a_2 r + b_2 r + c_2 r = pr,$$

and

$$\begin{aligned} S_{\triangle ABC} &= S_{\triangle AA_BI_A} + S_{\triangle AA_CI_A} - S_{\triangle BA_BI_A} - S_{\triangle BI_aI_A} - S_{\triangle CI_aI_A} - S_{\triangle CA_CI_A} = \\ &= pr_A - p_C r_A - p_B r_A = r_A p_A \end{aligned}$$

and the analogous formulas for the other escribed circles.

Thus, we obtain (together with formula (2)) the following representations for the area

$$S = a_2 h_A = b_2 h_B = c_2 h_C = pr = p_A r_A = p_B r_B = p_C r_C. \tag{8}$$

It provides another four pairs of dual variables: (r, p), (r_A, p_A) (r_B, p_B) (r_c, p_C).

The sides of the triangle are connected with the side partitions by the formulas (5), (6) and (7) in a very simple manner. By duality, we obtain easily relationships of the dual variables that are not so obvious, such as the formula $h_A = \frac{2r_B r_C}{r_B + r_C}$, a relationship between an altitude and two radii of the escribed circles. Replacing $a_2 \Leftrightarrow \frac{1}{h_A}$, $p_B \Leftrightarrow \frac{1}{r_B}$ and $p_C \Leftrightarrow \frac{1}{r_C}$, from $2a_2 = p_B + p_C$ (formula (4)) follows

$$\frac{2}{h_A} = \frac{1}{r_B} + \frac{1}{r_C} \ ,$$

or $h_A = \frac{2r_B r_C}{r_B + r_C}$. Analogously, can be derived the following relationships:

$$\begin{aligned} p = p_A + p_B + p_C &\Leftrightarrow \frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}, \\ p = a_2 + b_2 + c_2 &\Leftrightarrow \frac{1}{r} = \frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C}, \\ p_A = p - 2a_2 &\Leftrightarrow \frac{1}{r_A} = \frac{1}{r} - \frac{2}{h_A} = -\frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C}. \end{aligned}$$

4. Two pairs of dual quadruples. Using the last relationships, from Heron′s formula follows $\frac{1}{S^2} = \frac{1}{r r_A r_B r_C}$ and therefor

$$S^2 = p p_A p_B p_C = r r_A r_B r_C \ .$$

The product of the four values p, p_A, p_B, p_C is equal to the product of their dual values r, r_A, r_B, r_C. Each of the quadruples (p, p_A, p_B, p_C) and (r, r_A, r_B, r_c) consist of a value, which occurs only once in the triangle – namely, p and r – and three cyclical variables. Examining formula (7) – seven products, resulting the area – from an aesthetic point of view, it is immediately apparent that there is a product $S = xy$ missing. The unknowns x and y should be triangle sizes that exist only once and form with the cyclic values a_2, b_2, c_2 and h_A, h_B, h_C quadruples (x, a_2, b_2, c_2) and (y, h_A, h_B, h_c) similar to (p, p_A, p_B, p_C) and (r, r_A, r_B, r_c).

The value x should satisfy $x = \frac{S^2}{a_2b_2c_2}$. With the well known formula $4RS = abc$, where R is the circumradius, follows

$$x = \frac{S^2}{a_2b_2c_2} = \frac{8S^2}{abc} = \frac{2S}{R}$$

Thus,

$$S = \frac{R}{2} \cdot x.$$

Obviously, $y = \frac{R}{2} =: R_2$, half of the circumradius, should be one of the unknown.

The other length $x = \frac{S}{R_2}$ can be found in the triangle, too. But this is not so easy and a nice exercise. The answer is $\frac{S}{R_2} = u_H$ is the semiperimeter of the triangle formed by the foot points of the altitudes – the so-called altitude triangle. The value R_2 is not only the half of the circumradius, but also the circumradius of the altitude triangle. The circumcircle of the altitude triangle is a very famous circle in the triangle – may be the most important circle in a triangle at all. It is called Feuerbach′s circle (after the mathematician who first described it) or nine-point circle, because it passes through nine significant points. These are, besides the three foot points of the altitudes, the midpoints of the sides of the triangle and the midpoints of the upper parts of the altitudes (the line segments from each vertex of the triangle to the orthocenter).

Finally, we obtain the following relations

$$\begin{aligned} S &= R_2u_H = a_2h_A = b_2h_B = c_2h_C = pr = p_Ar_A = p_Br_B = p_Cr_C, \\ S^2 &= pp_Ap_Bp_C = rr_Ar_Br_C = R_2a_2b_2c_2 = u_Hh_Ah_Bh_C \ . \end{aligned}$$

These amazing formulas raise the question, are there some other dual quadruples in the triangle?

Recreaţii ... matematice

Numărul palindromic **918273645546372819** poate fi rezultatul unui calcul ce comportă doar două operaţii aritmetice cu numere ce se scriu cu cifrele 0 şi 1 (în baza de numeraţie 10)?

(Răspuns la pag. 28)