A new proof of the Blundon inequality

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Abstract. In this paper we present a new proof of the fundamental triangle inequality (Blundon's inequality) by reducing it to an analysis of the extremes of a convenient function.

Keywords: semiperimeter, inradius, circumradius, fundamental triangle inequality.

MSC 2010: 51M16.

In this Note, the fundamental triangle inequality or Blundon's inequality is treated as a problem of extremes of a convenient real function. On the same line the particular case of the acute triangles is detailed.

Let $\mathcal{C}(O,R)$ and $\mathcal{C}(I,r)$ be two circles verifying the condition $d^2 = R^2 - 2Rr$, where $d = OI$. Let be $A \in \mathcal{C}(O,R)$ and B, C be the intersection points of the tangents from A to the circle $\mathcal{C}(I,r)$ with the circle $\mathcal{C}(O,R)$. It is known that the straight line BC is tangent to the circle $\mathcal{C}(I,r)$. Therefore, the triangle ABC is inscribed in $\mathcal{C}(O,R)$ and it circumscribes the circle $\mathcal{C}(I,R)$. We shall use usual notations for triangles.

Let A_1, A_2 be the points in which the straight line OI intersects the circle $\mathcal{C}(O,R)$. If the point A takes the positions A_1, A_2 then the resulting triangles $A_1B_1C_1, A_2B_2C_2$ are isosceles. See Fig. 1.

Lemma 1. *(i) The lengths of the sides of the triangle $A_1B_1C_1$ are given by*

$$a_1 = 2\sqrt{R^2 - (r-d)^2}, b_1 = c_1 = \sqrt{2R(R+r-d)}, \tag{1}$$

while those of the sides of the triangle $A_2B_2C_2$ are given by

$$a_2 = 2\sqrt{R^2 - (r+d)^2}, b_2 = c_2 = \sqrt{2R(R+r+d)}. \tag{2}$$

(ii) The perimeter of the triangle $A_1B_1C_1$ is:

$$s_1 = \sqrt{\frac{(R+r-d)^3}{R-r-d}} = \sqrt{2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}}, \tag{3}$$

while that of the triangle $A_2B_2C_2$ is:

$$s_2 = \sqrt{\frac{(R+r+d)^3}{R-r+d}} = \sqrt{2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3}}. \tag{4}$$

Proof. (i) Let be $E = B_1C_1 \bigcap \mathcal{C}(I,r)$. See the Figs. 1 and 2. In the triangle B_1EO we have $B_1O^2 = B_1E^2 + EO^2$. It follows that $B_1E^2 = R^2 - (r-d)^2$, hence $a_1 = 2\sqrt{R^2 - (r-d)^2}$. In the triangle $A_1B_1A_2$ we have $A_1B_1^2 = AA_1.A_1E$. It follows that $b_1 = c_1 = \sqrt{2R(R+r-d)}$.

(ii) One similarly proceeds setting $E = B_2C_2 \bigcap \mathcal{C}(I,r)$.

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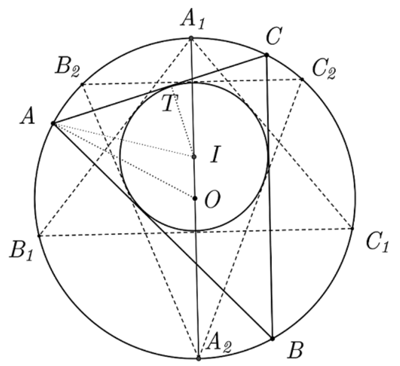

Fig. 1

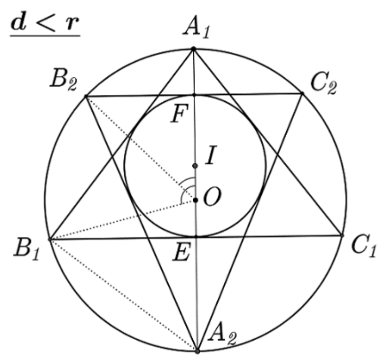

Fig. 2

Lemma 2. *Let be* $\alpha_1 = \mu(\widehat{B_1OA_1})$ *and* $\alpha_2 = \mu(\widehat{B_2OA_1})$. *We have*

$$\cos\alpha_1 = \frac{d-r}{R}, \cos\alpha_2 = \frac{d+r}{R}. \tag{5}$$

Proof. If $d < r$ (see Fig. 2), then $\cos\alpha_1 = -\cos\mu(\widehat{B_1OA_2}) = -\cos\mu(\widehat{B_1OE}) = -\frac{OE}{OB_1} = \frac{d-r}{R}$ and $\cos\alpha_2 = -\frac{OF}{OB_2} = \frac{d+r}{R}$. One similarly proceeds if $d \geq r$, see Fig. 3.

Theorem 1 (*Fundamental triangle inequality or Blundon's inequality*). *For any triangle ABC having the circumradius R, the inradius r and the semiperimeter s it is true that*

$$s_1 \leq s \leq s_2, \tag{6}$$

where

$$s_1 = \sqrt{\frac{(R+r-d)^3}{R-r-d}}, s_2 = \sqrt{\frac{(R+r+d)^3}{R-r+d}}. \tag{7}$$

The equality occurs in the left-side (resp. right-side) inequality if and only if the triangle ABC is isosceles having the sides length given by (1)*(resp.* (2)*).*

Proof. Let be $\alpha = \mu(\widehat{AOA_1}) \in [0,\pi]$ and $T = AC \bigcap \mathcal{C}(I,r)$. See the Fig. 1. In the triangle ATI we have $AI^2 = (s-a)^2 + r^2$ and applying the cosine law in the triangle AOI one obtains $AI^2 = R^2 + d^2 - 2Rd\cos\alpha$. It follows that

$$s - a = \sqrt{2R^2 - 2Rr - r^2 - 2Rd\cos\alpha}. \tag{8}$$

Going again to the triangle ATI we find $\tan\frac{A}{2} = \frac{r}{s-a}$. Thus we get

$$a = 2R\sin A = 2R.\frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} = 2R.\frac{2(s-a)r}{(s-a)^2+r^2}. \tag{9}$$

Using (8)and (9) in the identity $s = (s-a) + a$ one gets

$$s = f(\alpha), \tag{10}$$

where the function f is given by

$$f(\alpha) = \sqrt{2R^2 - 2Rr - r^2 - 2Rd\cos\alpha}.\frac{R + r - d\cos\alpha}{R - r - d\cos\alpha}, \alpha \in [0, \pi]. \tag{11}$$

The derivative of the function f is

$$f'(\alpha) = \frac{d\sin\alpha[Rd^2\cos^2\alpha + 2Rd(2r - R)\cos\alpha + R^3 - 4R^2 r + 3Rr^2 + 2r^3]}{(R - r - d\cos\alpha)^2\sqrt{2R^2 - 2Rr - r^2 - 2Rd\cos\alpha}}.$$

Since $R^3 - 4R^2 r + 3Rr^2 + 2r^3 = (R - 2r)(d^2 - r^2)$ the content in square brackets can be successively written in the forms:

$$\begin{aligned} &Rd^2\cos^2\alpha - 2d^3\cos\alpha + (R - 2r)(d^2 - r^2) = \\ &= \frac{d^2}{R}[R^2\cos^2\alpha - 2Rd\cos\alpha + d^2) - r^2] = \\ &= \frac{d^2}{R}(R\cos\alpha - d - r)(R\cos\alpha - d + r) = \\ &= Rd^2(\cos\alpha - \frac{d - r}{R})(\cos\alpha - \frac{d + r}{R}) = \\ &= Rd^2(\cos\alpha - \cos\alpha_1)(\cos\alpha - \cos\alpha_2). \end{aligned}$$

(The last equality follows from the Lemma 2).

Therefore, the derivative of the function f takes the form:

$$f'(\alpha) = \frac{Rd^3\sin\alpha(\cos\alpha - \cos\alpha_1)(\cos\alpha - \cos\alpha_2)}{(R - r - d\cos\alpha)^2\sqrt{2R^2 - 2Rr - r^2 - 2Rd\cos\alpha}}. \tag{12}$$

and it vanishes when α takes the values $0, \pi, \alpha_1, \alpha_2$. By the Lemma 2 we have the order $0 < \alpha_2 < \alpha_1 < \pi$ and (12) shows that $f'(\alpha) \geq 0$ if $\alpha \in [0, \alpha_2] \bigcup [\alpha_1, \pi]$ and $f'(\alpha) \leq 0$ if $\alpha \in [\alpha_2, \alpha_1]$. Consequently, the function f is increasing on the intervals $[0, \alpha_2], [\alpha_1, \pi]$ and is decreasing on the interval $[\alpha_2, \alpha_1]$.

Therefore, we have

$$\min[f(0), f(\alpha_1)] \leq s \leq \max[f(\alpha_2), f(\pi)]. \tag{13}$$

By a direct computation and noticing the Lemma 1 one finds

$$f(0) = \sqrt{\frac{(R + r - d)^3}{R - r - d}} = s_1, f(\pi) = \sqrt{\frac{(R + r + d)^3}{R - r + d}} = s_2. \tag{14}$$

On the other hand, by the very definition of s_1 and s_2 we have $s_1 = f(\alpha_1)$ and $s_2 = f(\alpha_2)$, hence $f(0) = f(\alpha_1), f(\pi) = f(\alpha_2)$ and so the equation (13) reduces to

$$f(0) \leq s \leq f(\pi), \tag{15}$$

which, by (14), is just the inequality to be proved. The cases in which the equality occurs were already established (see (15), (10) and (14)).

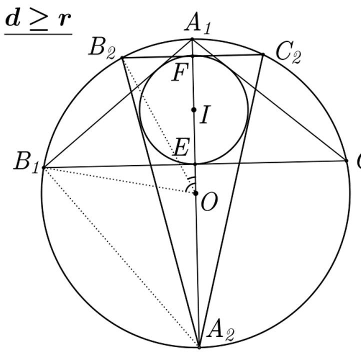

Fig. 3

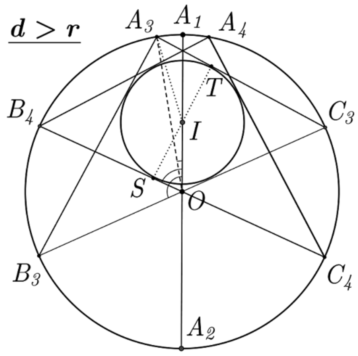

Fig. 4

In the following we will analysis the case $d > r$, equivalently $R > (\sqrt{2}+1)r$, that is the case when the point O is exterior to the circle $\mathcal{C}(I,r)$. The tangents through O to the circle $\mathcal{C}(I,r)$ intersect the circle $\mathcal{C}(I,R)$ in the points B_3, C_3, B_4, C_4. Next one construct the right triangles $A_3B_3C_3$ and $A_4B_4C_4$ inscribed in $\mathcal{C}(I,R)$ and circumscribing $\mathcal{C}(I,r)$, see the Fig. 4.

We set $\alpha_3 = \mu(\widehat{A_3OA_1}), \alpha_4 = \mu(\widehat{B_4OA_1}), \alpha_5 = \mu(\widehat{B_3OA_1})$.

Lemma 3. *The following sequence of inequalities*

$$0 < \alpha_3 < \alpha_2 < \alpha_4 < \alpha_1 < \alpha_5 < \pi$$

holds good.

Proof. Let us show that

$$1 > \cos\alpha_3 > \cos\alpha_2 > \cos\alpha_4 > \cos\alpha_1 > \cos\alpha_5 > -1.$$

By (5) we know $\cos\alpha_1, \cos\alpha_2$. In the triangle OSI we have $\sin\alpha_4 = \dfrac{r}{d}$, hence $\cos\alpha_4 = \sqrt{1-\frac{r^2}{d^2}}$. Similarly, $\sin\alpha_5 = \frac{r}{d}$, hence $\cos\alpha_5 = -\sqrt{1-\frac{r^2}{d^2}}$. By applying the cosine law in the triangle A_3OI and taking into account that $A_3I = r\sqrt{2}$ one obtains $\cos\alpha_3 = \dfrac{R^2-Rr-r^2}{Rd}$. Therefore, the above sequence of inequalities takes the form

$$1 > \frac{R^2-Rr-r^2}{Rd} > \frac{d+r}{R} > \sqrt{1-\frac{r^2}{d^2}} > \frac{d-r}{R} > -\sqrt{1-\frac{r^2}{d^2}} > -1$$

and now each inequality is easy to be checked.

Lemma 4. *The right triangle $A_3B_3C_3$ has the properties:*

(i) $s_3 = 2R + r$,

(ii) $a_3 = 2R, b_3 = R + r - \sqrt{R^2-2Rr-r^2}, c_3 = R + r + \sqrt{R^2-2Rr-r^2}$.

Proof. Since A_3TI is an isosceles right triangle we have $A_3T = IT$, hence $s_3 - a_3 = r$. So (i) holds good.

(ii) We have $\sum a_3 = 2s_3, \sum a_3b_3 = s_3^2 + r^2 + 4Rr, \prod a_3 = 2R + r$ and by (i) we get

$\sum a_3 = 4R + 2r, \sum a_3 b_3 = 4R^2 + 8Rr + 2r^2, \prod a_3 = 4Rr(2R - r)$. Thus, a_3, b_3, c_3 are the solutions of the equations

$$u^3 - (4R + 2r)u^2 + (4R^2 + 8Rr + 2r^2)u - 4Rr(2R - r) = 0.$$

This can be decomposed into the form

$$(u - 2R)([u^2 - 2(R + r)u + 4Rr + 2r^2] = 0,$$

and so (ii) quickly follows.

Lemma 5 (C. Ciamberlini, [**1**]). *A triangle is an acute triangle if and only if* $s > 2R + r$.

Theorem 2 (*Blundon's inequality in acute triangle*). *For any acute triangle* ABC *having the circumradius* R*, the inradius* r *and the semiperimeter* s *it is true that*

$$(16) \qquad s_1 \le s \le s_2, \quad if \quad 2 \le \frac{R}{r} < \sqrt{2} + 1,$$

$$(17) \qquad 2R + r \le s \le s_2, \quad if \quad \frac{R}{r} > \sqrt{2} + 1,$$

where s_1, s_2 *are given by* (7). *The equality occurs for the isosceles triangles* $A_1B_1C_1$ *and* $A_2B_2C_2$ *with the sides from the Lemma 1 and for the right triangle* $A_3B_3C_3$ *with the sides from the Lemma 4.*

Proof. It suffices to prove that in the case $d > r$ i.e. $\frac{R}{r} > \sqrt{2} + 1$ the inequalities (18) hold good. Indeed, since the triangles $A_3B_3C_3$ and $A_4B_4C_4$ are congruent, their perimeters are equal and by the Lemma 4 we have $s_3 = s_4 = 2R + r$. Consequently, $f(\alpha_3) = f(\alpha_4) = f(\alpha_5) = 2R + r$. In the proof of the Theorem 1 it was shown that f is increasing on $[0, \alpha_2], [\alpha_1, \pi]$. By Lemma 3 we have that $\alpha_3 \in [0, \alpha_2], \alpha_5 \in [\alpha_1, \pi]$. Thus $s = f(\alpha) \ge min[f(\alpha_3), f(\alpha_5)] = 2R + r$. Therefore, $s \ge 2R + r$, the equality holding for the right triangle having the sides given by the (ii) from the Lemma 4.

References

1. **C. Ciamberlini** - *Sulla condizione necessaria e suficiente affinchè un triangolo sia acutangolo, retangolo o ottusangolo*, Boll. Unione Mat. Ital., 5(1945), 37–41.

2. **D.S. Mitrinović, J.E. Pečarić, V. Volenec** - *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1989.

3. **S-H.Wu, Y-M. Chu** - *Geometric interpretation of Blundon's inequaliy and Ciamberlini's inequality*, J. of Ineq. Appl., 2014, 1–18.

Recreații ... matematice

Grigore Moisil este autorul unor memorabile aforisme. Iată două dintre ele:

Tot ce este gândire corectă este matematică sau susceptibilă de matematizare.
Fericirea se obţine prin integrarea unui soi de nimicuri.

Conform primului, al doilea aforism se poate *matematiza* (*modela matematic*). Însuşi Grigore Moisil indică operaţia prin care se poate realiza modelarea: *integrarea.* Găsiţi o rezolvare, fie ea umoristică!

Valeriu Braşoveanu, Bârlad

(*Rezolvare la p. 133*)