Several ways to develop an inequality

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Abstract. In this paper there are presented some developments of a given inequality. Detailed solutions are provided.

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MSC 2010: 51M16.

The starting point of this paper is the following double inequality proposed by **Daniel Sitaru** in *Art of Solving Problem*, December 23, 2016:

Let $a, b > 0$. Prove that

$$9 \le \left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) \le 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right).$$

In the literature there are at least six solutions of this problem. The following two are of interest for our purpose.

Solution 1 (**Soumava Chakraborty**, Calcutta, India).

$$\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right)\right) = 3 + \frac{4\sqrt{a}}{a+b} + \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2} + \frac{(a+b)^2}{4ab}$$
$$\le 3 + 2 + \frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} = 6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab}.$$

It suffices to prove that

$$6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \le 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right),$$

which is equivalent to

$$1 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \le \frac{2(a^2+b^2)}{ab}. \tag{1}$$

By the geometric-harmonic means inequality we have $\sqrt{ab} \ge \dfrac{2ab}{a+b}$, hence $\dfrac{a+b}{\sqrt{ab}} \le \dfrac{(a+b)^2}{2ab}$. It follows that

$$\frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} \le 1 + \frac{3(a+b)^2}{4ab}. \tag{2}$$

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Now (1) and (2) show that the right-side inequality will be proved if we prove that

$$\frac{3(a+b)^2+4ab}{4ab} \le \frac{2(a^2+b^2)}{ab}.$$

This is equivalent to $3a^2+3b^2+10ab \le 8a^2+8b^2$, or $5a^2+5b^2-10ab \ge 0$, which is simply $5(a-b)^2 \ge 0$.

Solution 2 (**Daniel Sitaru**). We recall the Schweitzer inequality:

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \le \frac{(m+M)^2n^2}{4mM},$$

where $x_1,\ldots,x_n \in [m,M], m>0$, and we use it with $n=3, m=x_1=\dfrac{2ab}{a+b}, x_2=\sqrt{ab}$ and $x_3=\dfrac{a+b}{2}=M$.

We directly get

$$A=\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right) \le 9\frac{\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)^2}{4ab}=$$

$$=\frac{9}{4ab}\left[\left(\frac{2ab}{a+b}\right)^2+2ab+\left(\frac{a+b}{2}\right)^2\right] \le \frac{9}{4ab}\left[\left(\sqrt{ab}\right)^2+2ab+\left(\frac{a+b}{2}\right)^2\right]=$$

$$=\frac{9}{4ab}\left[3ab+\left(\frac{a+b}{2}\right)^2\right]=\frac{9}{16ab}[14ab+(a^2+b^2)]=\frac{63}{8}+\frac{9}{16}\left(\frac{a}{b}+\frac{b}{a}\right).$$

Since $1 \le \dfrac{1}{2}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$, it follows that $\dfrac{23}{8} \le \dfrac{23}{16}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$ and $\dfrac{63}{8}+\dfrac{9}{16}\left(\dfrac{a}{b}+\dfrac{b}{a}\right) \le \dfrac{40}{8}+\dfrac{32}{16}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)=5+2\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$.

We notice that in the left-side inequality follows from the arithmetic-harmonic means inequality

$$E=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9.$$

The right-side of the above double inequality can be developed on various ways as we will show below.

Extension 1 (**Marin Chirciu**). *Let $a,b>0$. Prove that*

$$9 \le \left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right) \le 9-2n+n\left(\frac{a}{b}+\frac{b}{a}\right), n \ge \frac{3}{4}.$$

Proof. We denote $\dfrac{2ab}{a+b}=x$, $\sqrt{ab}=y$, $\dfrac{a+b}{2}=z$ and

$$E=\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right).$$

We have

$$(3)\qquad E=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right).$$

We prove that

$$(4)\ \left(\frac{x}{y}+\frac{y}{x}\right)\le 1+\frac{(a+b)^2}{4ab},\left(\frac{y}{z}+\frac{z}{y}\right)\le 1+\frac{(a+b)^2}{4ab},\left(\frac{z}{x}+\frac{x}{z}\right)=\frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2}.$$

Indeed, using and harmonic-geometric means inequality one obtains:

$$\frac{x}{y}+\frac{y}{x}=\frac{x^2+y^2}{xy}=\frac{\left(\frac{2ab}{a+b}\right)^2+ab}{\frac{2ab}{a+b}\cdot\sqrt{ab}}\le\frac{\left(\frac{2ab}{a+b}\right)^2+ab}{\frac{2ab}{a+b}\cdot\frac{2ab}{a+b}}=1+\frac{(a+b)^2}{4ab},$$

$$\frac{y}{z}+\frac{z}{y}=\frac{y^2+z^2}{yz}=\frac{ab+\left(\frac{a+b}{2}\right)^2}{\sqrt{ab}\cdot\frac{a+b}{2}}\le\frac{ab+\left(\frac{a+b}{2}\right)^2}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}}=1+\frac{(a+b)^2}{4ab},$$

$$\frac{z}{x}+\frac{x}{z}=\frac{x^2+z^2}{xz}=\frac{\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}}=\frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2}.$$

We look for an inequality having the form $E\le k+n\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$.

Taking into account (3) and (4) it remains to impose

$$3+\left[1+\frac{(a+b)^2}{4ab}\right]+\left[1+\frac{(a+b)^2}{4ab}\right]+\frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2}\le k+n\left(\frac{a}{b}+\frac{b}{a}\right)\Leftrightarrow$$

$$2\cdot\frac{(a+b)^2}{4ab}+\frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2}\le x-5+\frac{y(a^2+b^2)}{ab}\Leftrightarrow$$

$$3(a+b)^4+16a^2b^2\le 4(x-5)ab(a+b)^2+4y(a^2+b^2)(a+b)^2\Leftrightarrow$$

$$(5)\ (4n-3)a^4+(4k+8n-32)a^3b+(8k+8n-74)a^2b^2+(4k+8n-32)a^3b+(4n-3)b^4\ge 0.$$

In Horner's scheme we put the condition that 1 to be double root and we obtain $16k+32n-144=0$, $32k+64n-288=0$, where from $k+2n=9$.

It follows that (5) can be written as

$$(a-b)^2[(4n-3)a^2+(8n-2)ab+(4n-3)b^2]\ge 0.$$

The right parenthesis is positive if $4n-3\ge 0$, hence $n\ge\dfrac{3}{4}$. Equality holds if and only if $a=b$ and as we have noticed before $E\ge 9$. Thus the proof is complete.

Remark. The initial inequality is obtained for $n=2$. The value $n=\dfrac{3}{4}$ provides the sharpest inequality of this type.

Let us denote by $M_a=\dfrac{a+b}{2}$, $M_g=\sqrt{ab}$, $M_h=\dfrac{2ab}{a+b}$, $M_p=\sqrt{\frac{a^2+b^2}{2}}$ the arithmetic, geometric, harmonic and squared means, respectively.

The right-side inequality from the Extension 1 refers to $\{x,y,z\}=\{M_h,M_g,M_a\}\subset\{M_h,M_g,M_a,M_p\}$. We wonder if similar inequalities hold in the cases 2) $\{x,y,z\}=$

$\{M_g, M_a, M_p\}$, 3) $\{x, y, z\} = \{M_h, M_a, M_p\}$ and 4)$\{x, y, z\} = \{M_h, M_g, M_p\}$. The answer is positive. More precisely, the following extensions of the initial double inequality hold. All were proposed and proved by **Marin Chirciu**.

Extension 2. *Let $a, b > 0$. Prove that*

$$9 \leq \left(\frac{a+b}{2} + \sqrt{ab} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{2}{a+b} + \sqrt{ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \leq$$
$$\leq 9 - 2n + n\left(\frac{a}{b} + \frac{b}{a}\right), n \geq \frac{3}{4}.$$

Extension 3. *Let $a, b > 0$. Prove that*

$$9 \leq \left(\frac{a+b}{2} + \frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{2}{a+b} + \frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \leq$$
$$\leq 9 - 2n + n\left(\frac{a}{b} + \frac{b}{a}\right), n \geq \frac{3}{4}.$$

Extension 4. *Let $a, b > 0$. Prove that*

$$9 \leq \left(\sqrt{ab} + \frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{1}{\sqrt{ab}} + \frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \leq$$
$$\leq 9 - 2n + n\left(\frac{a}{b} + \frac{b}{a}\right), n \geq \frac{5}{4}.$$

The proofs of these inequalities are very similar to the proof of the inequality given as Extension 1. We give the proof of the Extension 4 and propose the reader to write-down the proofs of the Extensions 2 and 3 as an useful exercise.

Proof of Extension 4. We are in the case $\{x, y, z\} = \{M_g, M_h, M_p\}$. Using the inequalities $\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$, $\sqrt{ab} \geq \frac{2ab}{a+b}$, one easily finds

(6) $$\frac{x}{y} + \frac{y}{x} \leq 1 + \frac{(a+b)^2}{4ab}, \frac{y}{z} + \frac{z}{y} \leq \frac{(a+b)^2}{2ab} + \frac{4ab}{(a+b)^2}, \frac{z}{x} + \frac{x}{z} \leq 1 + \frac{a^2+b^2}{2ab}.$$

Next, we are looking for an inequality having the form $E \leq k + n\left(\frac{a}{b} + \frac{b}{a}\right)$. Based on (6) this is equivalent to

(7) $(4n-5)a^4 + (4k+8n-32)a^3b + (8k+8n-70)a^2b^2 + (4k+8n-32)a^3b + (4n-5)b^4 \geq 0$.

If in Horner's scheme one puts the condition that 1 to be double root one obtains $16k + 32n - 144 = 0$, $32k + 64n - 288 = 0$, wherefrom $k + 2n = 9$. It follows that (7) can be written as

$$(a-b)^2[(4n-5)a^2 + (4k+16n-42)ab + (16k+36n-149)b^2] \geq 0 \Leftrightarrow$$
$$(a-b)^2[(4n-5)a^2 + (8n-6)ab + (4n-5)b^2] \geq 0.$$

Notice that the right parenthesis is positive if $4n - 5 \geq 0$. The equality holds if and only if $a = b$. The proof is complete.

References

1. Daniel Sitaru – *Algebraic Phenomenon*, Paralela 45 Publishing House, Piteşti, 2017

2. *** – *Romanian Mathematical Magazine*, *http://www.ssmrmh.ro.*