Some properties of the straight lines which separate two parabolas

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Abstract. According to [**1**] a convex parabola and a concave one are separated by at least a straight line. In this paper we show that there are at most two straight lines that separate the said parabolas and are tangent to the both of them. Some interesting relations between the vertices of the parabolas and the tangent points are highlighted.

Keywords and phrases: parabolas, vertices, common tangents.

MSC 2010: 51M16.

A straight line separates the graphs of two functions if these graphs are located in different half-planes determined by this straight line. Here we are interested on the particular case of quadratic functions, that is on the case when a straight line separates two parabolas and especially on the case when the separating straight line is a common tangent to the given parabolas. In this particular case there exist some nice relations between the vertices of the parabolas and the tangent points.

We consider the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_1x^2 + b_1x + c_1$, where $a_1, b_1, c_1 \in \mathbb{R}$, $a_1 \neq 0$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = a_2x^2 + b_2x + c_2$, where $a_2, b_2, c_2 \in \mathbb{R}$, $a_2 \neq 0$.

The existence of the straight lines which separate the graphics of these two functions f and g is proved in case $a_1 > 0$, $a_2 < 0$ (or vice versa), in [**1**]. We are interested on how many of these straight lines, which are tangent to the graphs of f and g exist and what properties do they have.

Theorem 1. *If $a_1 > 0$ and $a_2 < 0$, then there are at most two straight lines of equation $y = mx + n$, which are tangent to the graphs of the functions f and g.*

Proof. Each one of the equations $f(x) = mx + n$ and $g(x) = mx + n$ has an unique solution if

$$(1)\qquad \begin{cases} (b_1 - m)^2 = 4a_1(c_1 - n) \\ (b_2 - m)^2 = 4a_2(c_2 - n) \end{cases}$$

Since the straight line $y = mx + n$ separates the two parabolas, we have $f(x) \geq mx+n \geq g(x), \forall x \in \mathbb{R}$ and $f(x) \geq g(x), \forall x \in \mathbb{R}$ implies $(b_1-b_2)^2 \leq 4(a_1-a_2)(c_1-c_2)$. By eliminating n from the system of equations (1), one deduces the following equation in m:

$$(2)\qquad (a_1 - a_2)m^2 - 2(a_1b_2 - a_2b_1)m + a_1b_2^2 - a_2b_1^2 + 4a_1a_2(c_1 - c_2) = 0.$$

Thus the existence of at most two real values for m is assured by : $4a_1a_2[(b_1 - b_2)^2 - 4(c_1 - c_2)(a_1 - a_2)] \geq 0$, due to $(b_1 - b_2)^2 \leq 4(c_1 - c_2)(a_1 - a_2)$ and $a_1a_2 < 0$, as well as of $c_2 < n < c_1$ yielded by (1). Concluding, there are at most two straight lines which are tangent to the given parabolas and which separate them.

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Theorem 2. *Let $y = mx + n$ be the straight line which is tangent to the graph of the function f in the point $T(t_1, f(t_1))$ and to the graph of the function g in the point $R(r_1, g(r_1))$. If $a_1 + a_2 = 0$ and $a_1 > 0$, we have:*

(i) $|t_1 - x_{V_1}| = |r_1 - x_{V_2}|$, where $V_1\left(x_{V_1}, -\frac{\Delta_1}{4a_1}\right)$ and $V_2\left(x_{V_2}, -\frac{\Delta_2}{4a_2}\right)$ are the vertices of the parabolas G_f and G_g (G stands for graph), with $V_1 \neq V_2$,

(ii) $V_1T \parallel V_2R$,

(iii) $V_1T = V_2R$.

Proof. (i) We the very definition we have $x_{V_1} = -\frac{b_1}{2a_1}$, $x_{V_2} = -\frac{b_2}{2a_2}$, and from the tangent condition one finds $t_1 = \frac{m-b_1}{2a_1}$ and $t_2 = \frac{m-b_2}{2a_2}$. Therefore, $|t_1 - x_{V_1}| = \left|\frac{m-b_1}{2a_1} + \frac{b_1}{2a_1}\right| = \frac{|m|}{2|a_1|}$ and $|t_2 - x_{V_2}| = \left|\frac{m-b_2}{2a_2} + \frac{b_2}{2a_2}\right| = \frac{|m|}{2|a_1|}$, so that (i) holds.

(ii) If the straight line V_1T does with the axis Ox the angle α, then

$$\tan\alpha = \left|\frac{f(t_1) - f(x_{V_1})}{t_1 - x_{V_1}}\right| = \left|\frac{mt_1 + n + \frac{\Delta_1}{4a_1}}{t_1 - x_{V_1}}\right| = \frac{|t_1 - x_{V_1}| \cdot |m|}{2|t_1 - x_{V_1}|} = \frac{|m|}{2}.$$

If the straight line V_2R does with the axis Ox the angle β, then

$$\tan\beta = \left|\frac{f(t_2) - f(x_{V_2})}{t_2 - x_{V_2}}\right| = \left|\frac{mr_1 + n + \frac{\Delta_2}{4a_2}}{t_2 - x_{V_2}}\right| = \frac{|t_2 - x_{V_2}| \cdot |m|}{2|t_2 - x_{V_2}|} = \frac{|m|}{2}.$$

Since the straight line $y = mx + n$ separates the given parabolas the α and β are acute angles. Therefore, $\alpha = \beta$ and so $V_1T \parallel V_2R$.

(iii) An obvious congruence of two right-angle triangles shows (iii).

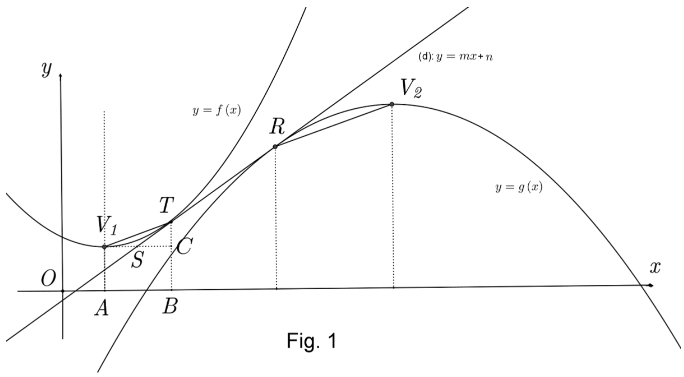

Fig. 1

Remark. In the conditions of Theorem 2, V_1TV_2R is a parallelogram.

Theorem 3. *Let be $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_1x^2 + b_1x + c_1$, with $a_1 > 0$ and the straight line $d : y = mx + n$, which is tangent to the graph of the function f in the point $T(t_1, f(t_1))$. If $C_1V_1 \parallel Ox$, $C_1V_1 \cap BT = \{C_1\}$ and $B \in Ox$ with $OB = |t_1|$, $t_1 > x_{V_1}$, then $[SV_1] \equiv [SC_1]$, where $d \cap C_1V_1 = \{S\}$.*

Proof. The intersection of the straight line (d) with the straight line C_1V_1 is the point $S\left(\frac{f(x_{V_1})-n}{m}, f(x_{V_1})\right)$. From the equality $(b_1 - m)^2 = 4a_1(c_1 - n)$, we deduce that $\frac{f(x_{V_1})-n}{m} = \frac{m}{4a_1} - \frac{b_1}{2a_1}$, so $SV_1 = \frac{|m|}{4a_1}$. On the other hand, $|t_1 - x_{V_1}| = \frac{|m|}{2a_1}$, so $2SV_1 = |t_1 - x_{V_1}| = AB$, where $A(x_{V_1}, 0)$ and $B(t_1, 0)$.

Theorem 4. *Let be again the parabolas G_f and G_g, the common tangent d separating them and T, R be the tangent points from Theorem 2. We consider the* *straight lines d_1 and d_2 which are parallel to d and are passing trough the points $T_1(t_1, f(t_1)+v)$ and $R_1(r_1, g(r_1)-v)$, respectively, for any real number $v>0$. Set $d_1 \cap G_f = \{M, N\}$ and $d_2 \cap G_f = \{U, V\}$. If $a_1 + a_2 = 0$, then $MN = UV$.*

Proof. The equation of the straight line d_1 is $y - (f(t_1)+v) = f'(t_1)(x-t_1)$ and its intersection points with G_f can be obtained from the equation $a_1x^2 - 2a_1t_1x + c_1 - a_1t_1^2 - b_1t_1 - c_1 - v + t_1(2a_1t_1 + b_1) = 0$ which is equivalent to $a_1x^2 - 2a_1t_1x + a_1t_1^2 - v = 0$ or $(x-t_1)^2 = \frac{v}{a_1} > 0$. The solutions of this equation are $x' = t_1 + \sqrt{\frac{v}{a_1}}$ and $x'' = t_1 - \sqrt{\frac{v}{a_1}}$, so that we have $M(x', f(x'))$ and $N(x'', f(x''))$. The length of the segment MN is
$MN = \sqrt{(x'-x'')^2 + (f(x') - f(x'')^2)}$
$= \sqrt{\frac{4v}{a_1}(1 + (f'(t_1))^2)}$.

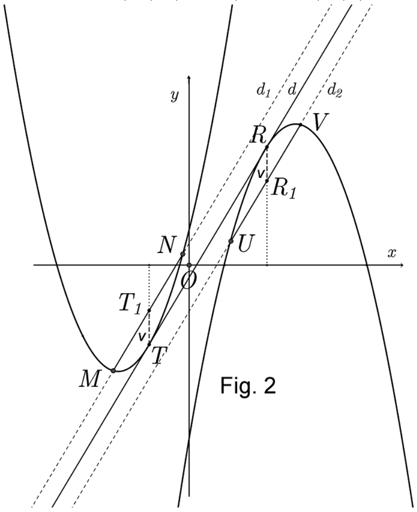

Fig. 2

We proceed similarly using the equation $y - (g(r_1) - v) = g'(r_1)(x - r_1)$ of d_2 in order to get $U(x_1', g(x_1'))$ and $V(x_1'', g(x_1''))$, where now $x_1' = r_1 + \sqrt{\frac{-v}{a_2}}$ and $x_1'' = r_1 - \sqrt{\frac{-v}{a_2}}$. The length of the segment UV is

$$UV = \sqrt{(x_1' - x_1'')^2 + (g(x_1') - g(x_1'')^2)} = \sqrt{\frac{4v}{a_1}(1 + (g'(r_1))^2)}.$$

Since the straight line (d) is tangent to the both parabolas in the points T and R, we have $f'(t_1) = g'(r_1)$. Hence the conclusion $MN = UV$ follows.

References

1. Dorin Andrica, Vasile Pop - *Problem 4*, Romanian Mathematical Olympiad, IX$^{\text{th}}$ grade, Final round, Bacău, 1995.

Recreaţii ... matematice

Radu a rezolvat următoarea problemă:

Arătaţi că oricum am înlocui spaţiile dintre cifre cu semnele de operaţii $+$ sau $-$ în

$$1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9,$$

după efectuarea calculelor nu putem obţine rezultatul 0.

Fire iscoditoare, a observat că dacă înlocuim doar şapte dintre spaţii cu unul dintre cele două semne, iar unul dintre spaţii îl ştergem (adică, de exemplu, în loc de $8-9$ considerăm numărul 89), atunci putem obţine rezultatul 0.

Daţi o soluţie problemei şi apoi daţi exemple care să confirme justeţea observaţiei lui Radu.

Titu Zvonaru, Comăneşti

(*Răspuns la p. 124*)