

## Some Properties of the Harmonic Quadrilateral<sup>1</sup>

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**Abstract.** In this article, we review some properties of the harmonic quadrilateral related to triangle simedians and to Apollonius circles.

**Keywords:** harmonic quadrilateral, simedian, Apollonius circle.

**MSC 2010:** 51M04.

**Definition 1.** A convex quadrilateral  $ABCD$  admitting a circumcircle and having the property  $AB \cdot CD = BC \cdot AD$  is called a *harmonic quadrilateral*.

**Definition 2.** A *triangle simedian* is the isogonal cevian of a triangle median.

**Proposition 1.** In the triangle  $ABC$ , the cevian  $AA_1$ ,  $A_1 \in (BC)$ , is a simedian if and only if  $\frac{BA_1}{A_1C} = \left(\frac{AB}{AC}\right)^2$ .

**Proposition 2.** In an harmonic quadrilateral, the diagonals are simedians of the triangles determined by two consecutive sides of a quadrilateral with its diagonal.

**Proof.** Let  $ABCD$  be an harmonic quadrilateral and  $\{K\} = AC \cap BD$  (Fig. 1). We prove that  $BK$  is simedian in the triangle  $ABC$ . From the similarity of the triangles  $ABK$  and  $DCK$ , we find that

$$(1) \quad \frac{AB}{DC} = \frac{AK}{DK} = \frac{BK}{CK}.$$

From the similarity of the triangles  $BCK$  și  $ADK$ , we conclude that

$$(2) \quad \frac{BC}{AD} = \frac{CK}{DK} = \frac{BK}{AK}.$$

From the relations (1) and (2), by division, it follows that

$$(3) \quad \frac{AB}{BC} \cdot \frac{AD}{DC} = \frac{AK}{CK}.$$

But  $ABCD$  is an harmonic quadrilateral; consequently,

$$\frac{AB}{BC} = \frac{AD}{DC};$$

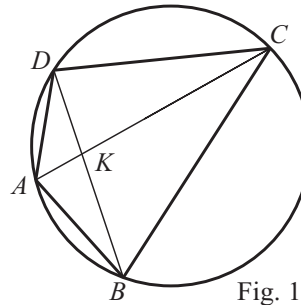


Fig. 1

<sup>1</sup>See also **A. Reisner** - *Quadrangle harmonique et nombres complexes*, this journal, 1/2014, 35-39.

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substituting this relation in (3), it follows that

$$\left(\frac{AB}{BC}\right)^2 = \frac{AK}{CK};$$

as shown by Proposition 1,  $BK$  is a simedian in the triangle  $ABC$ . Similarly, it can be shown that  $AK$  is a simedian in the triangle  $ABD$ , that  $CK$  is a simedian in the triangle  $BCD$ , and that  $DK$  is a simedian in the triangle  $ADC$ .

**Remark 1.** The converse of the Proposition 2 is proved similarly, i.e. we have:

**Proposition 3.** *If in a convex quadrilateral admitting a circumcircle a diagonal is a simedian in the triangle formed by the other diagonal with two consecutive sides of the quadrilateral, then the quadrilateral is an harmonic quadrilateral.*

**Remark 2.** From Propositions 2 and 3 above, it results a simple way to build an harmonic quadrilateral. In a circle, let a triangle  $ABC$  be considered; we construct the simedian of  $A$ , be it  $AK$ , and we denote by  $D$  the intersection of the simedian  $AK$  with the circle. The quadrilateral  $ABCD$  is an harmonic quadrilateral.

**Proposition 4.** *In a triangle  $ABC$ , the points of the simedian of  $A$  are situated at proportional distances to the sides  $AB$  and  $AC$ .*

**Proof.** We have the simedian  $AA_1$  in the triangle  $ABC$  (Fig. 2). We denote by  $D$  and  $E$  the projections of  $A_1$  on  $AB$ , and  $AC$  respectively. We get:

$$\frac{BA_1}{CA_1} = \frac{\text{Area}(\Delta ABA_1)}{\text{Area}(\Delta ACA_1)} = \frac{AB \cdot A_1D}{AC \cdot A_1E}.$$

Moreover, from Proposition 1, we know that  $\frac{BA_1}{A_1C} =$

$\left(\frac{AB}{AC}\right)^2$ . Substituting in the previous relation, we obtain that

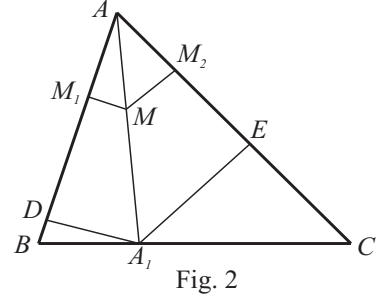
$$\frac{A_1D}{A_1E} = \frac{AB}{AC}.$$

On the other hand,  $A_1D = AA_1 \cdot \sin \widehat{BAA_1}$  and  $A_1E = AA_1 \cdot \sin \widehat{CAA_1}$ , hence

$$(4) \quad \frac{A_1D}{A_1E} = \frac{\sin \widehat{BAA_1}}{\sin \widehat{CAA_1}} = \frac{AB}{AC}.$$

If  $M$  is a point on the simedian and  $MM_1$  and  $MM_2$  are its projections on  $AB$ , and  $AC$  respectively, we have:  $MM_1 = AM \cdot \sin \widehat{BAA_1}$ ,  $MM_2 = AM \cdot \sin \widehat{CAA_1}$ , hence:

$$\frac{MM_1}{MM_2} = \frac{\sin \widehat{BAA_1}}{\sin \widehat{CAA_1}}.$$



Taking into account (4), we obtain that

$$\frac{MM_1}{MM_2} = \frac{AB}{AC}.$$

**Remark 3.** The converse of the property in the statement above is valid, meaning that, if  $M$  is a point inside a triangle, its distances to two sides are proportional to the lengths of these sides, then  $M$  the point belongs to the simedian of the triangle having the vertex joint to the two sides.

**Proposition 5.** In an harmonic quadrilateral, the point of intersection of the diagonals is located towards the sides of the quadrilateral to proportional distances to these sides.

The **Proof** of this statement relies on Propositions 2 and 4.

**Proposition 6 (R. Tucker).** The point of intersection of the diagonals of an harmonic quadrilateral minimizes the sum of squares of distances from a point inside the quadrilateral to the quadrilateral sides.

**Proof.** Let  $ABCD$  be an harmonic quadrilateral and  $M$  any point within. We denote by  $x, y, z, u$  the distances of  $M$  to the  $AB, BC, CD, DA$  sides of lengths  $a, b, c,$  and  $d$  (Fig. 3). Let  $S$  be the area of the quadrilateral  $ABCD$ . We have:  $ax + by + cz + du = 2S$ . Following Cauchy-Buniakowski-Schwarz Inequality, we get:

$$\begin{aligned} (a^2 + b^2 + c^2 + d^2) (x^2 + y^2 + z^2 + u^2) \\ \geq (ax + by + cz + du)^2, \end{aligned}$$

and it is obvious that

$$x^2 + y^2 + z^2 + u^2 \geq \frac{4S^2}{a^2 + b^2 + c^2 + d^2}.$$

We note that the minimum sum of squared distances is  $\frac{4S^2}{a^2 + b^2 + c^2 + d^2} = const.$  In Cauchy-Buniakowski-Schwarz Inequality, the equality occurs if

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{u}{d}.$$

Since  $\{K\} = AC \cap BD$  is the only point with this property, it ensues that  $M = K$ , so  $K$  has the property of the minimum in the statement.

**Definition 3.** We call *external simedian* of  $ABC$  triangle a cevian  $AA'_1$  corresponding to the vertex  $A$ , where  $A'$  is the harmonic conjugate of the point  $A_1$  – simedian's foot from  $A$  relative to points  $B$  and  $C$ .

**Remark 4.** In Fig. 4, the cevian  $AA_1$  is an internal simedian, and  $AA'_1$  is an

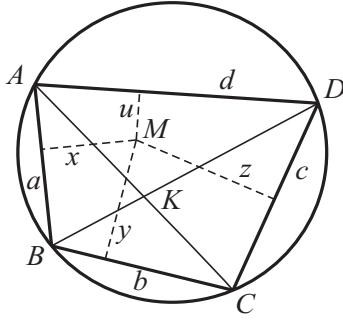


Fig. 3

external simedian.

We have  $\frac{A_1B}{A_1C} = \frac{A'_1B}{A'_1C}$ . In view of Proposition 1, we get that

$$\frac{A'_1B}{A'_1C} = \left(\frac{AB}{AC}\right)^2.$$

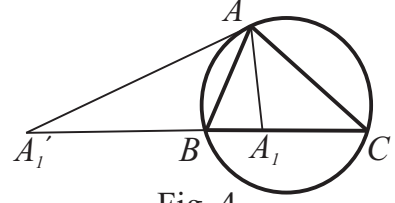


Fig. 4

**Proposition 7.** *The tangents taken to the extremes of a diagonal of a circle circumscribed to the harmonic quadrilateral intersect on the other diagonal.*

**Proof.** Let  $P$  be the intersection of a tangent taken in  $D$  to the circle circumscribed to the harmonic quadrilateral  $ABCD$  with  $AC$  (Fig. 5). Since triangles  $PDC$  and  $PAD$  are alike, we conclude that:

$$(5) \quad \frac{PD}{PA} = \frac{PC}{PD} = \frac{DC}{AD}.$$

From relations (5), we find that:

$$(6) \quad \frac{PA}{PC} = \left(\frac{AD}{DC}\right)^2.$$

This relationship indicates that  $P$  is the harmonic conjugate of  $K$  with respect to  $A$  and  $C$ , so  $DP$  is an external simedian from  $D$  of the triangle  $ADC$ .

Similarly, if we denote by  $P'$  the intersection of the tangent taken in  $B$  to the circle circumscribed with  $AC$ , we get:

$$(7) \quad \frac{P'A}{P'C} = \left(\frac{BA}{BC}\right)^2.$$

From (6) and (7), as well as from the properties of the harmonic quadrilateral, we know that  $\frac{AB}{BC} = \frac{AD}{DC}$ , which means that  $\frac{PA}{PC} = \frac{P'A}{P'C}$ , hence  $P = P'$ .

Similarly, it is shown that the tangents taken to  $A$  and  $C$  intersect at point  $Q$  located on the diagonal  $BD$ .

**Remark 5.** 1) The points  $P$  and  $Q$  are the diagonal poles of  $BD$  and  $AC$  in relation to the circle circumscribed to the quadrilateral.

2) From the previous proposition, it follows that in a triangle the internal simedian of an angle is consecutive to the external simedians of the other two angles.

**Proposition 8.** *Let  $ABCD$  be an harmonic quadrilateral inscribed in the circle of center  $O$  and let  $P$  and  $Q$  be the intersections of the tangents taken in  $B$  and  $D$ , respectively in  $A$  and  $C$  to the circle circumscribed to the quadrilateral. If  $\{K\} = AC \cap BD$ , then the orthocenter of triangle  $PKQ$  is  $O$ .*

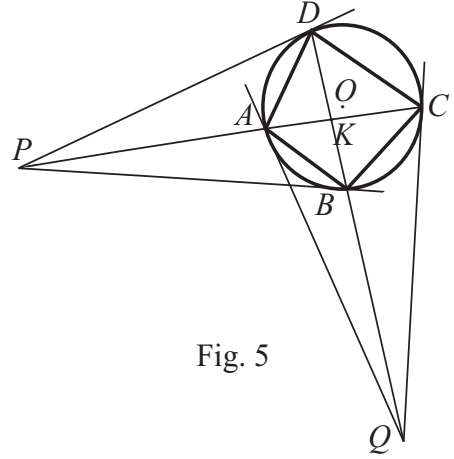


Fig. 5

**Proof.** From the properties of tangents taken from a point to a circle, we conclude that  $PO \perp BD$  and  $QO \perp AC$ . These relations show that in the triangle  $PKQ$ ,  $PO$  and  $QO$  are heights, so  $O$  is the orthocenter of this triangle.

**Definition 4.** The *Apollonius circle* related to the vertex  $A$  of the triangle  $ABC$  is the circle built on the segment  $[DE]$  in diameter, where  $D$  and  $E$  are the feet of the internal, respectively, external bisectors taken from  $A$  to the triangle  $ABC$ .

If the triangle  $ABC$  is isosceles with  $AB = AC$ , the Apollonius circle corresponding to vertex  $A$  is not defined.

**Proposition 9.** *The Apollonius circle relative to the vertex  $A$  of the triangle  $ABC$  has as center the feet of the external simedian taken from  $A$ .*

**Proof.** Let  $O_a$  be the intersection of the external simedian of the triangle  $ABC$  with  $BC$  (Fig. 6). Assuming that  $m(\widehat{B}) > m(\widehat{C})$ , we find that  $m(\widehat{EAB}) = \frac{1}{2} [m(\widehat{B}) + m(\widehat{C})]$ .  $O_aA$  being a tangent, we find that  $m(\widehat{O_aAB}) = m(\widehat{C})$ . From,  $m(\widehat{EAO_a}) = \frac{1}{2} [m(\widehat{B}) - m(\widehat{C})]$  and  $m(\widehat{AEO_a}) = \frac{1}{2} [m(\widehat{B}) - m(\widehat{C})]$  it results that  $O_aE = O_aA$ ; onward,  $EAD$  being a right angled triangle, we obtain  $O_aA = O_aD$ , hence  $O_a$  is the center of Apollonius circle corresponding to the vertex  $A$ .

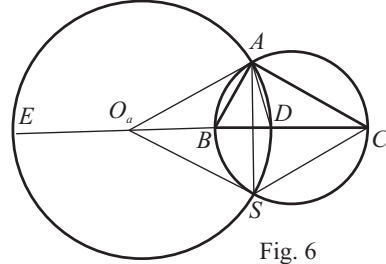


Fig. 6

**Proposition 10.** *Apollonius circle relative to the vertex  $A$  of the triangle  $ABC$  cuts the circle circumscribed to the triangle following the internal simedian taken from  $A$ .*

**Proof.** Let  $S$  be the second point of intersection of Apollonius circles relative to vertex  $A$  and the circle circumscribed to the triangle  $ABC$ . Because  $O_aA$  is tangent in  $A$  to the circle circumscribed, it results, for reasons of symmetry, that  $O_aS$  will be tangent in  $S$  to the circumscribed circle.

For triangle  $ACS$ ,  $O_aA$  and  $O_aS$  are external simedians; it results that  $CO_a$  is an internal simedian in the triangle  $ACS$ , furthermore, it results that the quadrilateral  $ABSC$  is a harmonic quadrilateral. Consequently,  $AS$  is the internal simedian of the triangle  $ABC$  and the property is proven.

**Remark 6.** From this, in view of Fig. 5, it results that the circle of center  $Q$  passing through  $A$  and  $C$  is an Apollonius circle relative to the vertex  $A$  for the triangle  $ABD$ . This circle (of center  $Q$  and radius  $QC$ ) is also an Apollonius circle relative to the vertex  $C$  of the triangle  $BCD$ .

Similarly, the Apollonius circles corresponding to vertexes  $B$  and  $D$  and to the triangles  $ABC$ , and  $ADC$  respectively, coincide.

**Proposition 11.** *In an harmonic quadrilateral, the Apollonius circles - associated with the vertexes of a diagonal and to the triangles determined by those vertexes to the other diagonal - coincide. Radical axis of the Apollonius circles is the right determined*

by the center of the circle circumscribed to the harmonic quadrilateral and by the intersection of its diagonals.

**Proof.** Referring to Fig. 5, we observe that the power of  $O$  towards the Apollonius circles relative to vertexes  $B$  and  $C$  of triangles  $ABC$  and  $BCU$  is  $OB^2 = OC^2$ . So  $O$  belongs to the radical axis of the circles.

We also have  $KA \cdot KC = KB \cdot KD$ , relatives indicating that the point  $K$  has equal powers towards the highlighted Apollonius circles.

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Al doilea autor va împlini în curând 60 de ani. Redacția revistei folosește acest prilej pentru a-i ura „MULȚI ANI !”, trăiți în sănătate, cu bucurii și noi și remarcabile succese. Dintr-un material amplu primit de redacție, selectăm un număr de aspecte care să dea contur personalității sale atipice, caracterizată prin energie explozivă, creativitate, diversitatea domeniilor abordate, prolificitate.

**Florentin Smarandache** s-a născut la 10 decembrie 1954 în Bălcești, jud. Vâlcea. Studiile, de la clasele primare la cele superioare, le-a făcut în România. A absolvit în 1979 secția informatică a Facultății de Științe a Universității din Craiova, ca șef de promoție.

În 1988 emigrează din motive politice. Din anul 2008 ocupă o poziție de profesor universitar la University of New Mexico (Gallup). În 1997 obține titlul de doctor în matematici, în domeniul teoriei numerelor, la Universitatea de Stat din Chișinău.

Are contribuții în matematică (teoria numerelor, geometrie, logică), fizică, filozofie, literatură (poeme, nuvele, povestiri, un roman, piese de teatru, eseuri, traduceri, interviuri) și artă (experimente în desene, picturi, colaje, fotografii, artă pe calculator).

În matematică a introdus gradul de negare al unei axiome ori teoreme (vezi geometriile smarandache, care pot fi parțial euclidiene și parțial neeuclidiene), multi-structurile (vezi n-structurile smarandache, unde o structură mai slabă conține insule de structuri mai puternice) și multi-spațiile (combinații de spații heterogene). Lucrările sale de teoria numerelor s-au bucurat de o anumită popularitate, fiind preluate și dezvoltate de matematicieni români și străini din multe țări. A propus extinderea probabilităților clasice și imprecise la „probabilitate neutrosifică”, ca un vector tridimensional ale cărui componente sunt submulțimi ale intervalului ne-standard ]-0,1+[. Împreună cu Jean Dezert extinde Teoria Dempster-Shafer la o nouă teorie de fuzionare a informației plauzibile și paradoxiste (numită Teoria Dezert-Smarandache).

În fizică a introdus paradoxuri cuantice și noțiunea de „nematerie”, formată din combinații de materie și antimaterie, ca o a treia formă posibilă de materie. A emis ipoteza că „nu există o barieră a vitezei în univers și se pot construi orice viteze”, pe baza căreia a propus o Teorie Absolută a relativității, care să nu producă dilatare a timpului sau contractare a spațiului.

În filozofie a introdus conceptul de „neutrosocie”, ca o generalizare a dialecticii lui Hegel, care stă la baza cercetărilor sale în matematică și economie, precum și „logica neutrosifică”, „mulțime neutrosifică”, „probabilitate neutrosifică”, „statistica neutrosifică”.

În literatură și artă a fondat în 1980 curentul de avangardă numit paradoxism, care constă în folosirea excesivă în creații a contradicțiilor, antitezelor, antinomiilor, oximoronilor, paradoxurilor. A fost nominalizat pentru premiul Nobel în literatură pe anul 2011.