An inequality on the medians of a triangle

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Abstract. In this Note a proof of the geometric inequality (1) below is provided.
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In this Note we shall give a proof of the following geometric inequality:

$$\frac{m_a^2+m_b^2+m_c^2}{2(m_am_b+m_bm_c+c_cm_a)-(m_a^2+m_b^2+m_c^2)}\le\frac{R}{2r}\tag{1}$$

with the usual notations in a triangle.

1. Preliminaries. First, we establish some results to be used later.

Let ABC be a triangle. Denote by ω its Brocard angle. The following properties of the angle ω are well-known:

$$(i)\ \ \operatorname{ctg}\omega=\frac{a^2+b^2+c^2}{4S},\quad (ii)\ \ \sin\omega=\frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}},\quad (iii)\ \ \omega\le\frac{\pi}{6},\tag{2}$$

and $\omega=\dfrac{\pi}{6}$ for the equilateral triangles. For details we refer to the Chapter XVII in [**2**].

Lemma 1. *In any triangle we have*

$$\frac{\sum a^2}{2\sum ab-\sum a^2}=t+\sqrt{t(t+1)},\tag{3}$$

where

$$t=\operatorname{ctg}^2\omega/(4\frac{R}{r}+1).$$

Proof. Taking into account (2(i)) as well as the formulas

$$\sum a^2=2(p^2-r^2-4Rr),\quad \sum ab=p^2+r^2+4Rr$$

one obtains that (3) is successively equivalent to:

$$\frac{\sum a^2}{4r(4R+r)}=\frac{(\sum a^2)^2}{16S^2}.\frac{r}{4R+r}+\frac{\sum a^2}{4S}\sqrt{\frac{r}{4R+r}(\frac{(\sum a^2)^2}{16S^2}\frac{r}{4R+r}+1)},$$

$$\frac{p}{4R+r}=\frac{\sum a^2}{4p(4R+r)}+\sqrt{\frac{r}{4R+r}+\frac{(p^2-r^2-4Rr)^2}{(4R+r)^2.4p^2}},$$

$$4p^2=\sum a^2+2\sqrt{4rp^2(4R+r)+(p^2-r^2-4Rr)^2},$$

$$(\sum a^2)^2-\sum a^2=2\sqrt{(p^2+r^2+4Rr)^2},$$

$$\sum ab=p^2+r^2+4Rr.$$

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Hence (3) holds.

Lemma 2. *In any triangle the following inequalities hold:*

$$\frac{1}{\sin^2\omega} \le \frac{4R^2-3Rr+6r^2}{4r^2} = \frac{R^2}{r^2} - \frac{3}{4}\frac{R}{r} + \frac{3}{2}, \tag{4}$$

$$\operatorname{ctg}^2\omega \le \frac{4R^2-3Rr+2r^2}{4r^2} = \frac{R^2}{r^2} - \frac{3}{4}\frac{R}{r} + \frac{1}{2}. \tag{5}$$

Proof. Taking into account that

$$\sum a^2b^2 = (\sum ab)^2 - 2abc\sum a = (p^2+r^2+4Rr)^2 - 16Rrp^2$$

one gets

$$\frac{1}{\sin^2\omega} = \frac{(p^2+r^2+4Rr)^2-16rp^2}{4r^2p^2}. \tag{6}$$

It is clear that (4) follows from the double inequality that will be proved below.

$$\frac{1}{\sin^2\omega} \le \frac{p^2+3r^2-7Rr}{4r^2} \le \frac{4R^2-3Rr+6r^2}{4r^2}. \tag{7}$$

Indeed, according to (6), the first inequality (7) reduces to $(p^2+r^2+4Rr)^2 - 16rp^2 \le p^2(p^2+3r^2-7Rr)$ and this in turn becomes $r^3+16R^2r+8Rr^2 \le p^2(r+R)$. Since by the second Gerretsen′s inequality $16Rr-5r^2 \le p^2$ [**3**], it suffices to show that $r^3+16R^2r+8Rr^2 \le (r+R)(16Rr-5r^2)$. By some algebra this inequality reduces to the Euler inequality $2r \le R$. Thus the first inequality (7) holds.

The second inequality (7) easily follows by using the first Gerretsen′s inequality: $p^2 \le 4R^2+4Rr+3r^2$ [**3**]. Indeed, $p^2+3r^2-7Rr \le (4R^2+4Rr+3r^2)+3r^2-7Rr = 4R^2-3Rr+6r^2$. Concluding, the proof of the inequality (4) is complete.

The identity $\operatorname{ctg}^2\omega = \dfrac{1}{\sin^2\omega} - 1$ help us to easily derive the inequality (5) from (4).

Remark. A remarkable consequence of the preceding results is the well-known inequalities

$$2 \le \frac{1}{\sin\omega} \le \frac{R}{r}. \tag{8}$$

(the left-side inequality follows from (2(iii)) and the right-side inequality follows from (4): $\dfrac{1}{\sin^2\omega} \le \dfrac{4R^2-3r(R-2r)}{4r^2} \le \dfrac{4R^2}{4r^2}$ etc.). In [**1**] one finds a refinement of the right-side inequality (8).

2. Proof of the inequality (1). Let $A_mB_mC_m$ be the triangle whose sides have the lengths respectively equal to the lengths m_a, m_b, m_c of the medians of the triangle ABC and let R_m, r_m, S_m, ω_m be entities easy to understand associated to it.

A well-known and very useful property says that $\omega = \omega_m$. Indeed, by (2(i)) we have $\operatorname{ctg}\omega_m = \dfrac{\sum m_a^2}{4S_m} = \dfrac{\frac{3}{4}\sum a^2}{4.\frac{3}{4}S} = \operatorname{ctg}\omega$ and (2(iii)) implies $\omega = \omega_m$.

Applying the Lemma 1 to the triangle $A_mB_mC_m$ one obtains

$$\frac{\sum m_a^2}{2\sum m_a m_b - \sum m_a^2} = t_m + \sqrt{t_m(t_m+1)},$$

where

$$t_m = \operatorname{ctg}^2\omega_m/(4\frac{R_m}{r_m}+1) = \operatorname{ctg}^2\omega/(4\frac{R_m}{r_m}+1).$$

Therefore, the inequality (1) reduces to

$$\sqrt{t_m(t_m+1)} \le \frac{R}{2r} - t_m. \tag{9}$$

We need to show that $\dfrac{R}{2r} - t_m \ge 0$ or, equivalently, $\operatorname{ctg}^2\omega \le \dfrac{R}{2r}\left(4\dfrac{R_m}{r_m}+1\right)$ or

$$\frac{1}{\sin^2\omega} \le 2\frac{R}{r}.\frac{R_m}{r_m} + \frac{1}{2}\frac{R}{r} + 1.$$

The last inequality is true since from (8) it follows that $\dfrac{R}{r} \ge \dfrac{1}{\sin\omega}$ and $\dfrac{R_m}{r_m} \ge \dfrac{1}{\sin\omega}$ and it is obvious that we have $\dfrac{1}{\sin^2\omega} \le 2\dfrac{1}{\sin\omega}\dfrac{1}{\sin\omega} + \dfrac{1}{2}\dfrac{1}{\sin\omega} + 1$.

Squaring, (9) can be written in the following equivalent forms:

$$t_m^2 + t_m \le \frac{1}{4}\frac{R^2}{r^2} - \frac{R}{r}t_m + t_m^2 \Leftrightarrow \left(\frac{R}{r}+1\right)t_m \le \frac{1}{4}\frac{R^2}{r^2}.$$

Thus the inequality (1) is equivalent to

$$\operatorname{ctg}^2\omega \le \frac{1}{4}\left(4\frac{R_m}{r_m}+1\right)\left[\frac{R^2}{r^2}/\left(\frac{R}{r}+1\right)\right]. \tag{10}$$

Case 1. Assume that $\dfrac{R}{r} \ge \dfrac{R_m}{r_m}$. Obviously, we have also

$$\frac{R^2}{r^2}/\left(\frac{R}{r}+1\right) \ge \frac{R_m^2}{r_m^2}/\left(\frac{R_m}{r_m}+1\right).$$

Thus we can write

$$\frac{1}{4}\left(4\frac{R_m}{r_m}+1\right)\left[\frac{R^2}{r^2}/\left(\frac{R}{r}+1\right)\right] \ge \frac{1}{4}\left(4\frac{R_m}{r_m}+1\right)\left[\frac{R_m^2}{r_m^2}/\left(\frac{R_m}{r_m}+1\right)\right]. \tag{11}$$

On the other hand, according to (5), we have

$$\operatorname{ctg}^2\omega = \operatorname{ctg}^2\omega_m \le \frac{R_m^2}{r_m^2} - \frac{3}{4}\frac{R_m}{r_m} + \frac{1}{2}. \tag{12}$$

By (10),(11) and (12) it follows that the inequality (10) is true if the following inequality holds:

$$\frac{R_m^2}{r_m^2} - \frac{3}{4}\frac{R_m}{r_m} + \frac{1}{2} \le \frac{1}{4}\left(4\frac{R_m}{r_m} + 1\right)\left[\frac{R_m^2}{r_m^2} / \left(\frac{R_m}{r_m} + 1\right)\right]. \tag{13}$$

Setting $\dfrac{R_m}{r_m} = u$, (13) takes the form

$$\left(u^2 - \frac{3}{4}u + \frac{1}{2}\right)(u+1) \le \frac{1}{4}(4u+1)u^2. \tag{14}$$

Next, (14) reduces to $u \ge 2$, that is $R_m \ge 2r_m$ (Euler). Thus, the inequality (1) is true in the considered case.

Case 2. Assume that $\dfrac{R}{r} \le \dfrac{R_m}{r_m}$. Then

$$\frac{1}{4}\left(4\frac{R_m}{r_m} + 1\right)\left[\frac{R^2}{r^2} / \left(\frac{R}{r} + 1\right)\right] \ge \frac{1}{4}\left(4\frac{R}{r} + 1\right)\left[\frac{R^2}{r^2} / \left(\frac{R}{r} + 1\right)\right]$$

and from (5) it follows that

$$\operatorname{ctg}^2 \omega \le \frac{R^2}{r^2} - \frac{3}{4}\frac{R}{r} + \frac{1}{2}.$$

The inequality (10), hence and (1) will be true if we show that

$$\frac{R^2}{r^2} - \frac{3}{4}\frac{R}{r} + \frac{1}{2} \le \frac{1}{4}(4\frac{R}{r} + 1)\left[\frac{R^2}{r^2} / \left(\frac{R}{r} + 1\right)\right]. \tag{15}$$

We notice that (15) is just the inequality (13) written for the triangle ABC. Thus it holds good and the inequality (1) is completely proved.

References

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