An application of Pell′s equation

Gheorghe IUREA[1]

Abstract. In this Note it is shown that the number 121 is the only square number with the property that the product of its figures is a prime number.

Keywords: prime number, square number, Pell equation.

MSC 2010: 11D09.

The main purpose of this Note is to solve the following problem as well as a problem related to it. The notions and the results to be used below can be found in [**1**], [**2**], [**3**], [**4**].

Problem 1. *Determine the square numbers with the property that the product of their figures is a prime number.*

Solution. If the number $\overline{a_1 a_2 ... a_n}$ is a square number and the product $a_1 \cdot a_2 \cdot ... \cdot a_n$ of its figures is a prime number then only one of its figures can be $2, 3, 5$ or 7 and the other ones are 1.

The following fact is well-known and easy to be checked.

Lemma. *If the number* $\overline{a_1 a_2 ... a_k}$ *is square number and* a_k *is an odd figure, then* a_{k-1} *is an even figure.*

Therefore, it does not exist square numbers whose all figures to be odd. The figures of the number $\overline{a_1 a_2 ... a_n}$ can be 1 and only one equal to 2. Observe that the case $a_n = 2$ is not possible. Thus, our problem reduces to determine all squares having the form $\overline{11...121}$ (n figures).

If $d = \overline{11...121}$ is a square number, then the number $9d = \overline{100...089} = 10^n + 89$ is also a square number. Let us determine the natural numbers n, m such that

$$10^n + 89 = m^2. \tag{1}$$

If $n = 2t$ the equation (1) can be written in the form $(m - 10^t)(m + 10^t) = 89$ and it follows that $m - 10^t = 1$ and $m + 10^t = 89$, a contradiction. Hence n is odd, $n = 2t + 1$. In this case we write the equation (1) in the form

$$m^2 - 10 \cdot \left(10^t\right)^2 = 89. \tag{2}$$

Therefore, we may regard the pair $(m, 10^t)$ as solution of the Diophantine equation

$$a^2 - 10b^2 = 89. \tag{3}$$

We seek to express the solutions of the equation (3) as functions of the solutions of the Pell equation:

$$x^2 - 10y^2 = 1. \tag{4}$$

[1] Profesor, Liceul Teoretic ,,Dimitrie Cantemir", Iaşi; *iurea_g@yahoo.com*

It is known that the natural solutions of the equation (4) are the pairs (x_k, y_k), where

(5)
$$x_k = \frac{1}{2}\left[\left(19+6\sqrt{10}\right)^k + \left(19-6\sqrt{10}\right)^k\right],$$
$$y_k = \frac{1}{2\sqrt{10}}\left[\left(19+6\sqrt{10}\right)^k - \left(19-6\sqrt{10}\right)^k\right], \quad k \in \mathbb{N}.$$

Therefore, we have:

(6)
$$x_k = C_k^0\, 19^k + C_k^2\, 19^{k-2}\, 6^2\, 10 + ...$$

(7)
$$y_k = C_k^1\, 19^{k-1}\, 6 + C_k^3\, 19^{k-3}\, 6^3\, 10 + ...$$

Let (a, b), $a, b > 0$, a solution of the equation (3). Then

(8)
$$A_p + B_p\sqrt{10} = \left(a + b\sqrt{10}\right)\left(19 - 6\sqrt{10}\right)^p,\ p \in \mathbb{N}$$

is a strictly decreasing sequence that converges to zero. By a direct computation it comes out that for every $p \in \mathbb{N}$ the pair (A_p, B_p) is a solution of the equation (3).

Since $A_0 + B_0\sqrt{10} = a + b\sqrt{10} > 1$, there exists an unique $k \in \mathbb{N}$ such that

$$A_{k+1} + B_{k+1}\sqrt{10} < 1 \le A_k + B_k\sqrt{10}.$$

From (8) it follows: $A_{k+1} + B_{k+1}\sqrt{10} = \left(A_k + B_k\sqrt{10}\right)\left(19 - 6\sqrt{10}\right) < 1$. Hence $A_k + B_k\sqrt{10} < 19 + 6\sqrt{10} < 38$.

On the other hand, since (A_k, B_k) is a solution of (3), we have:

$$\left(A_k + B_k\sqrt{10}\right)\left(A_k - B_k\sqrt{10}\right) = 89.$$

From here we infer that $2 < A_k - B_k\sqrt{10} \le 89$. By adding side by side the last two inequalities one obtains $3 < 2A_k < 127$, hence $A_k \in \{2, 3, ..., 63\}$.

Taking into account the equation

$$B_k^2 = \frac{A_k^2 - 89}{10},$$

we conclude that the solutions (A_k, B_k) are only the pairs $(27, -8)$ and $(33, -10)$.

Using (8) we get

(9)
$$a + b\sqrt{10} = \left(27 - 8\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^k$$

or

(10)
$$a + b\sqrt{10} = \left(33 - 10\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^k.$$

As from (5) we infer that $\left(19+6\sqrt{10}\right)^k = x_k + y_k\sqrt{10}$, it follows that the solutions (a, b) of the equation (3) are given by

$$a = 27x_k - 80y_k, \qquad b = 27y_k - 8x_k \tag{11}$$

or

$$a = 33x_k - 100y_k, \qquad b = 33y_k - 10x_k. \tag{12}$$

Thus, there exists $k \in \mathbb{N}$ such that

$$10^t = 27y_k - 8x_k \tag{13}$$

or

$$10^t = 33y_k - 10x_k. \tag{14}$$

From (6) it follows that $x_k \equiv 1 \pmod 3$, and from (14) it follows $x_k \equiv -1 \pmod 3$, hence $1 \equiv -1 \pmod 3$, a contradiction. Therefore, (14) does not hold true.

For $t \geq 3$, from (13) it follows that $y_k \vdots 8$. Taking into account (7), we infer $C_k^1\, 19^{k-1}\, 6 \vdots 8$, hence $k \vdots 4$, i.e. $k = 4q, q \in \mathbb{N}$.

For q even, x_k is odd and $y_k \vdots 16$. Using (13), for $t \geq 4$ we arrive at a contradiction. Thus q is odd.

Taking x_k, y_k modulo 19, from (6) and (7) we obtain: $x_k \equiv C_k^k\, 6^k\, 10^{\frac{k}{2}} \equiv 6^{4q}\, 10^{2q} \equiv 360^{2q} \equiv (-1)^{2q} \equiv 1 (\bmod 19)$ and $y_k \equiv 0 (\bmod 19)$. From (13), it follows that

$$10^t \equiv 11 (\bmod 19). \tag{15}$$

Calculating the powers of 10 modulo 19, from (15) we find that $t = 18s + 6$, $s \in \mathbb{N}$.

The initial equation (1) reduces to

$$10^{36s+13} + 89 = m^2. \tag{16}$$

According to Fermat′s Little Theorem we have $10^{12} \equiv 1 (\bmod 13)$. Thus, from (16) we get $m^2 \equiv 10 + 89 (\bmod 13)$, that is $m^2 \equiv 8 (\bmod 13)$, an equation that has no solutions. Therefore, the equation (16) has no solutions for $t \geq 4$.

It remains to consider $t \leq 3$. It comes out that only $t = 1$ is convenient, hence $n = 3$, and the square number searched for is 121.

Concluding, the Problem 1 is solved.

The next problem, related to the Problem 1 is much easier.

Problem 2. *Determine square numbers that have n figures ($n \geq 2$) equal to* 1 *and one equal to a figure a.*

Solution. According to Lemma above, do not exist square numbers having only odd figures. Thus, we have no solutions for a odd. If a is 0, 4 or 8, then a can not be the last figure. Contrary, the searched numbers are divisible by 2 but are not divisible by 4. Again by Lemma above, the figure a will be the penultimate figure. As $\overline{11...1a1} = \overline{11...1000} + \overline{1a1} = \mathcal{M}8 + 5$ and the square of an odd number is $\mathcal{M}8 + 1$, it follows that there are no solutions for $a \in \{\underline{0, 4, 8}\}$.

For $a = 6$, we have to analyse the numbers $\overline{11...16}$ and $\overline{11...161}$. We have: $\overline{11...16} = 4 \cdot \overline{277...79} = 4 \cdot (\mathcal{M}4 + 3)$ hence it is not a square number and $\overline{11...161}$ is $\mathcal{M}11 + 6$ or $\mathcal{M}11 + 7$ and also it is not a square number since it has no the form $\mathcal{M}11 + r$ with $r \in \{0, 1, 3, 4, 5, 9\}$.

For $a = 2$, reasoning as in the preceding case we find the square number 121.

Concluding, there exists only a square number with the property of the statement of the Problem 2.

References

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ERATĂ

I. *Recreaţii Matematice, nr. 1/2016.* La pp. 39-40, în articolul *O remarcă asupra verificării rădăcinilor unor ecuaţii iraţionale* – **A. Vernescu,** numerotarea formulelor s-a făcut greşit, cu numerele de la 3 la 9. Se va face modificarea $n \longrightarrow n - 2, n \in \overline{3, 9}$, pentru ca trimiterile din text să fie adresate corect formulelor.

II. *Recreaţii Matematice, nr. 2/2016.* La pp. 124-127, în articolul *Obţinerea relaţiei de recurenţă pentru un anumit şir de integrale definite*– **A. Vernescu,** numerotarea formulelor s-a făcut greşit cu numerele de la 19 la 30. Corectarea se face prin modificarea $n \longrightarrow n - 18, n \in \overline{19, 30}$.

Recreații ... matematice

La 27 ianuarie 2017, o persoană respectabilă a fost întrebată ce vârstă are.

– *M-am născut într-un an pătrat perfect şi vârsta mea este un pătrat perfect.*

– *Mulţi ani fericiţi!*

Aflaţi anul naşterii şi vârsta persoanei.

Neculai Stanciu, Titu Zvonaru

(*Răspuns la p. 37*)