A new method to solve inequalities

Marius DRĂGAN[1]***, Neculai STANCIU***[2]

Abstract. In this paper we present a new method to find the best constant k for the inequality $F(s,R,r)\geq k$ and for its reverse.

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The purpose of this article is to give a new and simple method to find the best constant k for the inequality

$$F(s,R,r)\geq k, \tag{1}$$

as well as for its reverse

$$F(s,R,r)\leq k, \tag{2}$$

where F is a homogeneous function of degree zero and monotone with respect to s.

Theorem (fundamental triangle inequality or Blundon′s inequality). *For any triangle ABC the inequalities*

$$s_1\leq s\leq s_2 \tag{3}$$

hold, where s_1 and s_2 are the semiperimeters of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ that have the same circumradius R and inradius r as the triangle ABC and their sides are given as follows:

$$a_1=2\sqrt{(R+r-d)(R-r+d)},\quad b_1=c_1=\sqrt{2R(R+r-d)},$$
$$a_2=2\sqrt{(R+r+d)(R-r-d)},\quad b_2=c_2=\sqrt{2R(R+r+d)},$$

where $d=\sqrt{R^2-2Rr}$.

We treat the inequality (1) only. The inequality (2) can be treated similarly.

If F is decreasing in the argument s, it follows that $F(s,R,r)\geq F(s_2,R,r)$, hence k is the best constant in (1) if and only if it is the best constant in the inequality

$$F(s_2,R,r)\geq k. \tag{4}$$

We denote

$$t=\sqrt{(R+r+d)(R-r-d)},\quad x=\sqrt{\frac{\sqrt{2R}-\sqrt{R-r-d}}{\sqrt{2R}+\sqrt{R-r-d}}}\in(0,1). \tag{5}$$

[1]Profesor, Colegiul Tehnic ,,Mircea cel Bătrân", Bucureşti; *marius.dragan2005@yahoo.com*
[2]Profesor, Şcoala Gimnazială ,,George Emil Palade", Buzău; *stanciuneculai@yahoo.com*

A straightforward computation yields

$$a_2 = 2t, b_2 = c_2 = \frac{(1+x^2)t}{1-x^2}, s_2 = \frac{2t}{1-x^2}, r_2 = r = xt, R_2 = R = \frac{(1+x^2)^2 t}{4x(1-x^2)}. \tag{6}$$

Then the inequality (4) takes the form

$$F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right) \geq k, \quad \forall x \in (0,1).$$

Consequently, the best constant k will be the $\inf_{x\in(0,1)} f(x)$, where

$$f(x) = F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right), \quad x \in (0,1). \tag{7}$$

In the case that F is increasing in the argument s, it follows that $F(s,R,r) \geq F(s_1,R,r)$, hence k is the best constant in (1) if and only if it is the best constant in the inequality

$$F(s_1, R, r) \geq k. \tag{8}$$

We denote

$$\tau = \sqrt{(R+r-d)(R-r+d)}, \quad x = \sqrt{\frac{\sqrt{2R} - \sqrt{R-r-d}}{\sqrt{2R} + \sqrt{R-r-d}}} \in (0,1), \tag{9}$$

and by a straightforward computation one obtains

$$a_1 = 2\tau, b_1 = c_1 = \frac{(1+x^2)\tau}{1-x^2}, s_1 = \frac{2\tau}{1-x^2}, r_1 = r = x\tau, R_1 = R = \frac{(1+x^2)^2\tau}{4x(1-x^2)}. \tag{10}$$

Then the inequality (8) can be written in the form

$$F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right) = f(x) \geq k, \forall x \in (0,1),$$

so the best constant k is the $\inf\limits_{x\in(0,1)} f(x)$.

In conclusion, whether F increases or decreases with respect to s, the best constant k in the inequality (1) is the infimum of the function f defined by (7).

Applications

I. *Find the best constant k for which the inequality*

$$\begin{aligned}&(x^2+y^2+z^2)(x^2+y^2+z^2-xy-yz-zx) \geq \\ &\geq k(x+y+z)(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2-6xyz)\end{aligned} \tag{11}$$

is true for any $x, y, z \geq 0$.

In [**1**], *Marian Tetiva* has solved this problem using the algebraic norming method. We present a new solution.

Solution. Let

$$u=\frac{x}{\sqrt{\Sigma xy}},\quad v=\frac{y}{\sqrt{\Sigma xy}},\quad w=\frac{z}{\sqrt{\Sigma xy}}. \tag{12}$$

Then, the inequality (11) is equivalent to

$$(u^2+v^2+w^2)(u^2+v^2+w^2-1)\geq k(u+v+w)(u+v+w-9uvw), \tag{13}$$

where $uv+vw+wu=1$.

Since the function $\operatorname{tg}\frac{x}{2}$ is bijective on $(0,\pi)$, it follows that there exists a triangle ABC such that

$$u=\operatorname{tg}\frac{A}{2}, v=\operatorname{tg}\frac{B}{2}, w=\operatorname{tg}\frac{C}{2}.$$

It is well-known that

$$\sum\operatorname{tg}\frac{A}{2}=\frac{4R+r}{s},\quad \prod\operatorname{tg}\frac{A}{2}=\frac{r}{s},\quad \sum\operatorname{tg}^2\frac{A}{2}=\frac{(4R+r)^2}{s^2}-2. \tag{14}$$

Thus the inequality (13) becomes

$$\left[\frac{(4R+r)^2}{s^2}-2\right].\frac{(4R+r)^2-3s^2}{(4R+r)(4R-8r)}\geq k. \tag{15}$$

This is an inequality of the form (1). It is obvious that $F(s,R,r)$ – the left hand side of (15) – is an homogeneous of degree zero and decreasing function in s (is product of two decreasing functions in s). By a tedious computation one obtains that the function f associated to it as in (7) is $f(x)=\dfrac{9x^4-2x^2+1}{12x^4+4x^2}$ and that its derivative is $f'(x)=\dfrac{15x^4-6x^2-1}{2x^3(3x^2+1)^2}$.

The roots of the equation $f'(x)=0$ are $x_{1,2}=\pm\sqrt{\dfrac{1}{9}+\dfrac{2}{9}\sqrt{\dfrac{2}{3}}}, x_1>x_2$. So, $k=\inf\limits_{x\in(0,1)}f(x)=f(x_1)=\sqrt{6}-2$, i.e. $k=\sqrt{6}-2$.

II. *Find the best constant k such that*

$$(x+y)^2(y+z)^2(z+x)^2-64x^2y^2z^2\geq k[xyz(x+y+z)^3-27x^2y^2z^2], \forall x,y,z>0. \tag{16}$$

This problem was proposed and left as an exercise by *Marian Tetiva* in [**1**]. Here we give our solution.

Solution. We proceed as in the previous problem to bring (16) to a geometric inequality of the form (1). With the notations (12) the inequality (16) becomes

$$\sum u.\sum uv-64(\prod u)^2\geq k\prod u\left[(\sum u)^3-27\prod u\right],$$

and this transforms by (14) to

$$\frac{16R^2-64r^2}{r[\frac{(4R+r)^3}{s^2}-27r]}\geq k,$$

that is an inequality of the form (1) with F increasing with respect to s. According to (7), for f and its derivative we find

$$f(x)=\frac{4(7x^4-10x^2-1)}{15x^4-14x^2-1},\quad f'(x)=\frac{32x(13x^4+4x^3-1)}{(15x^4-14x^2-1)^2}.$$

The equation $f'(x)=0$ has the real roots

$$x_0=0,\quad x_{1,2}=\pm\sqrt{\frac{\sqrt{17}}{13}-\frac{2}{13}}\simeq 0.404,\quad x_1>x_0,\quad x_1^4=\frac{1}{13}(1-4x_1^2).$$

Therefore,

$$k=\inf_{x\in(0,1)}f(x)=f(x_1)=\frac{4(79x_1^2+3)}{121x_1^2-1}=\frac{\sqrt{17}+23}{8}\simeq 3.3903,$$

that is $k=\frac{\sqrt{17}+23}{8}$ is the best constant for the inequality (16).

III. *Find the best constants for which the inequalities*
(i) $a^2+b^2+c^2\leq 4\sqrt{3}S+k[(a-b)^2+(b-c)^2+(c-a)^2]$,
(ii) $a^2+b^2+c^2\geq 4\sqrt{3}S+k[(a-b)^2+(b-c)^2+(c-a)^2]$
are true in any triangle with area S.

It is well-known that the best constant in (i) is $k=3$ and in (ii) this is $k=1$ i.e. when (ii) reduces to Hadwiger-Finsler inequality [**2**]. We prove these results using our method.

Solution. (i) Using the identities $\sum ab=s^2+r^2+4Rr, \sum a^2=2(s^2-r^2-4Rr), S=rs$ the inequality (i) can be written in the form

$$F(s,R,r)=\frac{s^2-2\sqrt{3}rs-r^2-4Rr}{s^2-3r^2-12Rr}\leq k$$

or

$$F(s,R,r)=1+\frac{8Rr+2r^2-2\sqrt{3}rs}{s^2-3r^2-12Rr}\leq k.$$

We have $8Rr+2r^2-2\sqrt{3}rs\geq 0$ because of Finsler's inequality $\sqrt{3}s\leq 4R+r$. From Gerretsen's inequality, $s^2\geq 16rR-5r^2$, we get $s^2\geq 3r^2+12Rr$. Therefore, F is a decreasing function in s since it is a product of two positive decreasing functions. By a direct calculation one finds

$$f(x)=\left(\frac{x+\sqrt{3}}{\sqrt{3}x+1}\right)^2,\quad f'(x)=-\frac{4(x+\sqrt{3})}{(\sqrt{3}x+1)^3}, x\in(0,1).$$

So, $f(x)$ is a decreasing function. Hence, $k = \sup_{x\in(0,1)} f(x) = f(0) = 3$.

(ii) The best constant for (ii) is $k = \inf_{x\in(0,1)} f(x) = f(1) = 1$.

In the following we shall use our method in order to obtain some refinements of Gerretsen′s inequality.

IV. *Find the best positive constant k such that the inequality*

$$a^2 + b^2 + c^2 + k\frac{r^2}{s^2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \le 8R^2 + 4r^2 \tag{17}$$

is true in any triangle ABC.

Solution. First, the inequality (17) can be written in the form (1) as follows:

$$F(s, R, r) = \frac{4R^2 + 3r^2 + 4Rr - s^2}{s^2 - 3r^2 - 12Rr} \cdot \frac{s^2}{r^2} \ge k.$$

We consider the function $h : (0, \infty) \mapsto \mathbb{R}$ given by

$$h(u) = \frac{(4R^2 + 3r^2 + 4Rr)u - u^2}{u - (3r^2 + 12Rr)^2},$$

whose derivative is

$$h'(u) = \frac{-u^2 + 2(3r^2 + 12Rr)u - (4R^2 + 3r^2 + 4Rr)(3r^2 + 12Rr)}{(u - 3r^2 - 12Rr)^2}.$$

The equation of degree two in u from the numerator has the discriminant $\Delta = -16(3r^2+12Rr)(R-2r)$ and it is obvious that $\Delta \le 0$. Hence $h'(u) \le 0$. Consequently, F is a decreasing function in s.

The best constant k is the infimum of the function

$$f(x) = F\left(\frac{2}{1-x^2}, \frac{(1+x^2)^2}{4x(1-x^2)}, x\right) = \frac{1}{x^4}.$$

Thus $k = \inf_{x\in(0,1)} \frac{1}{x^4} = 1$.

Concluding, the inequality

$$a^2 + b^2 + c^2 + \frac{r^2}{s^2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \le 8R^2 + 4r^2 \tag{18}$$

is the best inequality of the type (17). It is obvious that (18) is a refinement for the right side of the Gerretsen inequality.

V. *Find the best positive constant k such that the inequalities*

(i) $a^2 + b^2 + c^2 + k \cdot \frac{r^2}{R^2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \le 8R^2 + 4r^2$ *and*

(ii) $a^2+b^2+c^2 \geq 24Rr - 12r^2 + k.\frac{r}{R}[(a-b)^2+(b-c)^2+(c-a)^2]$,
are true in any triangle ABC.

Solution. (i). We write (i) in the form

$$k \leq \frac{4R^2+3r^2+4Rr-s^2}{s^2-3r^2-12Rr}.\frac{R^2}{r^2} = \left[\frac{4R(R-2r)}{s^2-3r^2-12Rr}-1\right].\frac{R^2}{r^2} = F(s,R,r).$$

Observe that F is a decreasing function in s. The best constant is $k = \inf_{x\in(0,1)} f(x)$, where $f(x) = \frac{(1+x^4)^4}{4x^4}, x \in (0,1)$. Since $f'(x) = \frac{(x^2-1)(x^2+1)^3}{x^5} < 0$, it follows that $k = f(1) = 4$. Concluding, the inequality

$$(19) \qquad a^2+b^2+c^2+4\frac{r^2}{R^2}[(a-b)^2+(b-c)^2+(c-a)^2] \leq 8R^2+4r^2$$

is valid in any triangle. It is another refinement of the right side of the Gerretsen inequality.

(ii) The inequality (ii) takes the following form:

$$k \leq \frac{s^2-16Rr+5r^2}{s^2-3r^2-12Rr}.\frac{R}{r} = \left[1-\frac{4R(R-2r)}{s^2-3r^2-12Rr}\right].\frac{R}{r} = F(s,R,r),$$

with F decreasing in s. By a direct computation one finds:

$$f(x) = \frac{(1+x^2)^2}{4(1-x^2)}, \quad f'(x) = \frac{x(x^2+1)(3-x^2)}{2(1-x^2)^2}, x \in (0,1).$$

Therefore, f is an increasing function on $(0,1)$. We get $k = \inf_{x\in(0,1)} f(x) = f(0) = \frac{1}{4}$.

Finally, in any triangle ABC we have the inequality

$$(20) \qquad a^2+b^2+c^2 \geq 24Rr-12r^2+\frac{1}{4}\frac{r}{R}[(a-b)^2+(b-c)^2+(c-a)^2],$$

which represents a refinement for the left side of the Gerretsen inequality.

References

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