About some identities and inequalities for complex numbers

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Abstract. In this paper we prove a general identity. By particularization, we obtain known identities and inequalities.

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Theorem 1. *For any* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$*, the identity*

$$\gamma\delta|\alpha z_1 + \beta z_2|^2 - \alpha\beta|\gamma z_1 + \delta z_2|^2 = (\alpha\delta - \beta\gamma)(\alpha\gamma|z_1|^2 - \beta\delta|z_2|^2) \tag{1}$$

holds.

Proof. We note $z_k = x_k + iy_k$, where $x_k, y_k \in \mathbb{R}$, $k \in \{1, 2\}$. Then the identity (1) is equivalent with

$$\gamma\delta[(\alpha x_1 + \beta x_2)^2 + (\alpha y_1 + \beta y_2)^2] - \alpha\beta[(\gamma x_1 + \delta x_2)^2 + (\gamma y_1 + \delta y_2)^2]$$
$$= (\alpha\delta - \beta\gamma)[\alpha\gamma(x_1^2 + y_1^2) - \beta\delta(x_2^2 + y_2^2)],$$

an identity which can be easily verified.

Application 1. *If* $\alpha = \gamma = \delta = 1$ *and* $\beta = -1$*, then we obtain the following well-known identity:*

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2) \tag{2}$$

for any $z_1, z_2 \in \mathbb{C}$.

Remark 1. Let z_1, z_2, $z_1 + z_2$ be the corresponding affixes of the points A, B respectively C. Then, the geometrical interpretation of the identity (2) is the following metrical relation in a parallelogram: $OC^2 + AB^2 = OA^2 + AC^2 + CB^2 + BO^2$ ([**6**]).

Application 2. *If* $\alpha = \beta = 1$, $\gamma = y$ *and* $\delta = -x$*, then we obtain the Bergström identity*

$$(x + y)(y|z_1|^2 + x|z_2|^2) - xy|z_1 + z_2|^2 = |yz_1 - xz_2|^2 \tag{3}$$

for any $x, y \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$ ([**2**]).

Remark 2. If in Bergström identity $x = y = 1$, then we obtain the identity (2).

Application 3. *From* (3) *it follows that*

$$(x + y)(y|z_1|^2 + x|z_2|^2) - xy|z_1 + z_2|^2 \geq 0 \tag{4}$$

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for any $x, y \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$. *The equality holds if and only if* $yz_1 = xz_2$. *If* $x, y > 0$ *and we denote* $\frac{1}{y(x+y)} = a_1$, $\frac{1}{x(x+y)} = a_2$, *from (4) we obtain*

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} \geq \frac{|z_1 + z_2|^2}{a_1 + a_2} \tag{5}$$

for any $z_1, z_2 \in \mathbb{C}$ *and* $a_1, a_2 > 0$, *which is a well known Bergström Inequality* ([**3**]).

Application 4. *Let* $a, b, c, m \in \mathbb{C}$, $b \neq c$, *where* m *verifies* $|c - m| + |m - b| = |c - b|$. *From this equation it results that* $m = \frac{|c-m|}{|c-b|} b + \frac{|m-b|}{|c-b|} c$. *Considering* $\alpha = \left|\frac{c-m}{c-b}\right|$, $\beta = -\left|\frac{m-b}{c-b}\right|$, $\gamma=\delta=\sqrt{|c-b|}$, $z_1=a-b$ *and* $z_2=c-a$ *in relation* (1), *we obtain*

$$|a - b|^2|c - m| + |c - a|^2|m - b| - |a - m|^2|c - b| = |c - b||m - b||c - m|. \tag{6}$$

Remark 3. Let a, b, c, m be the corresponding affixes of the points A, B, C, respectively M. From the relation $|c - m| + |m - b| = |c - b|$ verified by m it results that $M \in (BC)$. Then we can write the equation (6) under the form

$$AB^2 \cdot MC + AC^2 \cdot MB - AM^2 \cdot BC = MB \cdot MC \cdot BC.$$

The above relation is the well-known Stewart relation.

Application 5. *In* (1) *we consider* $z_1 = a_2$, $z_2 = b$, $\alpha = \beta = 1$, $\gamma, \delta \in \mathbb{R}^*$, *what leads to*

$$|a + b|^2 - \frac{1}{\gamma\delta} |\gamma a + \delta b|^2 = \left(1 - \frac{\gamma}{\delta}\right) |a|^2 + \left(1 - \frac{\delta}{\gamma}\right) |b|^2. \tag{7}$$

If $\gamma\delta < 0$ *and* $c = -\dfrac{\gamma}{\delta}$, *from* (7) *we obtain the inequality*

$$|a + b|^2 \leq (1 + c)|a|^2 + \left(1 + \frac{1}{c}\right) |b|^2 \tag{8}$$

for any $a, b \in \mathbb{C}$ *and* $c > 0$.

Remark 4. The inequality (8) is called the Bohr inequality ([**4**]).

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